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One-dimensional physics
Problem set 3

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In the lecture, we have expressed the creation operators for chiral fermions in a quantum wire as

$$r^\dagger(x) = \lim_{\alpha\to 0} \frac{U_r^\dagger}{\sqrt{2\pi\alpha}} e^{i\Phi_r(x)}, \tag{1}$$

where $r = R, L$ denotes the chirality (i.e. direction of motion) of the fermions, $1/\alpha$ is a large-momentum cutoff, U_r denotes a Klein factor, and where the bosonized field $\Phi_r(x)$ satisfies the commutator

$$\lim_{\alpha\to 0}\left[\Phi_r(x), \Phi_{r'}(x')\right] = \lim_{\alpha\to 0}\lim_{L\to\infty} \delta_{r,r'} \ln\left(\frac{1-e^{2\pi[\alpha+i\hat{r}(x-x')]/L}}{1-e^{2\pi[\alpha-i\hat{r}(x-x')]/L}}\right) = \delta_{r,r'}\, i\pi\hat{r}\,\mathrm{sgn}(x-x') \tag{2}$$

with $\hat{R} = +1$ and $\hat{L} = -1$, and where L denotes the length of the wire (which is assumed to be very large). We furthermore decomposed these fields as

$$\Phi_r(x) = \Phi_r^+(x) + \Phi_r^-(x) \qquad \text{with} \qquad [\Phi_r^+(x), \Phi_{r'}^-(x')] = \delta_{r,r'} \ln\left(1 - e^{-2\pi[\alpha+i\hat{r}(x-x')]/L)}\right), \tag{3}$$

where $\Phi_r^+(x)$ contains only bosonic creation operators (for the bosons associated with density waves), and with $\Phi_r^-(x) = [\Phi_r^+(x)]^\dagger$.

1. Point splitting and the fermionic anticommutator **2 Points**

In the fermionic representation, the chiral operators satisfy the canonical anticommutator

$$\{r^\dagger(x), r'(x')\} = \delta_{r,r'}\delta(x-x'). \tag{4}$$

a) 1 Point

Using $U_r^\dagger U_r = 1 = U_r U_r^\dagger$, the fact that Klein factors acting on different species anticommute, and the fact that the Klein factors and bosonized fields commute with each other, convince yourself that the anticommutator is satisfied for $x \neq x'$ by directly using Eq. (1), as well as Eq. (2) in the limit $\alpha \to 0$. Which problem do you encounter for $x = x'$, and how can you understand this problem?

b) 1 Point

To calculate the commutator directly, one can instead use a protocol known as "point splitting", which can be understood as an operator product expansion (OPE). Considering a general product $\mathcal{O}_i(x)\mathcal{O}_j(x')$, where $\mathcal{O}_k(x)$ are operators labelled by an index k and depending on a coordinate x, the OPE extracts the most singular contribution to the product of operators as $x \to x'$. Technically speaking, this most singular contribution is the leading term of the Laurent series

$$\mathcal{O}_i(z)\mathcal{O}_j(z') = \sum_k C_{ijk} \frac{\mathcal{O}_k(z')}{(z-z')^{\Delta_i+\Delta_j-\Delta_k}}\ , \tag{5}$$

where $\mathcal{O}_{i,j,k}(z)$ are operators that depend on complex numbers z and z', while C_{ijk} are complex constants, and $\Delta_{i,j,k}$ are the so-called scaling-dimensions of $\mathcal{O}_{i,j,k}(z)$. In this spirit, the OPE of the anticommutator can be evaluated by keeping α and $(x-x')$ finite. More precisely, the anticommutator, expressed in bosonized fields, can be evaluated for fixed finite α and $(x-x')$ upon taking the limit $L \to \infty$, and by then expanding the bosonized fields Φ in $(x-x')$ to obtain the leading operator term for the OPE. Why can this expansion only be done for after normal ordering the expression? Hint: you can use

$$\lim_{\alpha\to 0}\frac{1}{\pi}\frac{\alpha}{\alpha^2+(x-x')^2}=\delta(x-x'). \tag{6}$$

2. Point splitting and the fermionic density **2 Points**

Use the same procedure as in the first exercise to calculate the bosonized expression for the density. To this end, start from the fermionic normal ordered density

$$^*_*\rho_r(x)^*_* = \lim_{x'\to x}\left(r^\dagger(x)r(x')-\langle 0|r^\dagger(x)r(x')|0\rangle\right), \tag{7}$$

where $|0\rangle$ is the vacuum. Then use the bosonization identity in Eq. (1), Taylor-expand the normal-ordered bosonized fields to first order in $(x-x')$, and finally take the limit $\alpha\to 0$ for fixed $(x-x')$. Hint: remembering the definition of $\Phi_r(x)$ in terms of bosonic operators and the average particle number δN_r, you can easily check that $\langle 0|\partial_x\Phi_r(x)|0\rangle = 0$.

3. Point splitting and the kinetic energy **2 Points**

As a final application of the point splitting scheme, we now want to analyze the expression of fermionic normal-ordered Hamiltionian for the kinetic energy, which is given by

$$H_{\text{kin.}} = \int dx\,\left(R^\dagger(x)(-i\partial_x)R(x)-L^\dagger(x)(-i\partial_x)L(x)-\langle 0|R^\dagger(x)(-i\partial_x)R(x)-L^\dagger(x)(-i\partial_x)L(x)|0\rangle\right), \tag{8}$$

with $|0\rangle$ still denoting the vacuum. To find the bosonized expression for the kinetic energy, start from the point-split expression, and expand the bosonized fields in $(x-x')$ after their normal ordering. Hint: you may find it helpful to expand the bosonized fields around $(x+x')/2$, and to then expanding the exponentials to order $(x-x')^2$. Also, note that the most simple variant of the point splitting procedure given in exercise two does not give rise to a Hermitian expression, which needs to be corrected in the end.