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One-dimensional physics
Problem set 5

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1. Conductance quantization in Luttinger liquids **7 Points**

On the first exercise sheet, we showed that a non-interacting quantum wire with n right-moving and n left-moving modes has a conductance of $G = n\,e^2/h$. This simple calculation is in agreement with experimental observations, even for interacting quantum wires, but was for a long time at odds with the predictions of Luttinger liquid theory. This discrepancy was only solved in 1995 by three seminal works of Maslov and Stone, Ponomarenko, and Safi and Schulz, who thus clarified an important aspect of Luttinger liquid calculations in real systems.

a) **1 Point**

The basis for all subsequent calculation is the Kubo formula for the electric conductance, which gives the current in a quantum wire in linear response to an applied electric field. We thus start by recalling the general framework of Kubo formulas by considering a 1D system described by an initial Hamiltonian H_0, to which a time-dependent perturbation $H_1(t)$ is applied. More precisely, $H_1(t)$ describes the coupling of so-called forces $F_j(x,t)$ (which are simple functions, not operators) to observables $\mathcal{O}_j(x,t)$ as

$$H_1(t) = \int dx \sum_j F_j(x,t)\,\mathcal{O}_j(x,t). \tag{1}$$

Assuming that the perturbation is switched on at a time t_0, use the time evolution in the interaction picture to derive the Kubo formula

$$\langle \mathcal{O}_k(x,t)\rangle \approx \langle \mathcal{O}_k(x,t)\rangle_0 + \int_{t_0}^{\infty} dt'\ \left(-i\theta(t-t')\langle[\mathcal{O}_k(x,t), H_1(t')]\rangle_0\right), \tag{2}$$

where $\langle\ \rangle_0$ denotes the expectation value with respect to the unperturbed system.

b) **1 Point**

The retarded correlation function of two observables $\mathcal{O}_k(x,t)$ and $\mathcal{O}_j(x',t')$ is defined as

$$C^{\mathrm{R}}_{kj}(x-x', t-t') = -i\theta(t-t')\langle[\mathcal{O}_k(x,t), \mathcal{O}_j(x',t')]\rangle_0, \tag{3}$$

which has the Fourier transform

$$C^{\mathrm{R}}_{kj}(x-x',\omega) = \lim_{\eta\to 0}\int_{-\infty}^{\infty} dt\, e^{i(\omega+i\eta)t}\, C^{\mathrm{R}}_{kj}(x-x',t). \tag{4}$$

Furthermore, we recall that the Fourier transform of an operator is given by

$$\mathcal{O}_k(x,\omega) = \int dt\, e^{i\omega t}\,\mathcal{O}_k(x,t). \tag{5}$$

Show that for $t_0 \to -\infty$, in which case the transient behaviour associated with the switching-on of $H_1(t)$ can be neglected, one finds

$$\langle \mathcal{O}_k(x,\omega)\rangle \approx \langle \mathcal{O}_k(x,\omega)\rangle_0 + \int dx' \sum_j F_j(x',\omega)\, C^{\mathrm{R}}_{kj}(x-x',\omega). \tag{6}$$

c) **1 Point**

We are now in particular interested to the response of a quantum wire to an applied electric field $\mathbf{E}(x)$. Starting with the Hamiltonian H_0 of free, spinless one-dimensional electrons, show that minimal coupling to external electromagnetic fields amounts to the Hamiltonian $H = H_0 + H_{\rm EM}$ with $H_{\rm EM} = \int dx\,(\rho\,\Phi - j_{\rm tot}\,A_x)$, where $\mathbf{A}$ is the vector potential, Φ the scalar potential, ρ is the charge density, and where the current $j_{\rm tot}$ is the sum of a "paramagnetic" contribution j, and $1/2$ times a "diamagnetic" current j^A with

$$j(x) = \frac{e}{2mi}\left(c^\dagger(x)[\partial_x c(x)] - [\partial_x c^\dagger(x)]c(x)\right) \quad \text{and} \quad j^A(x) = -\frac{e^2}{m}A_x\, c^\dagger(x)c(x), \tag{7}$$

where $c(x)$ annihilates an electron at position x. For an applied electric field $\mathbf{E}$ that is constant along the wire, you can use a gauge such that $\Phi = 0$ to show that the linear-response current flowing due to the applied electric field is given by

$$j_{\rm tot}(x,\omega) = \langle j_{\rm tot}(x,\omega)\rangle_0 + \int dx'\,\frac{1}{-i\omega}E_x(x',\omega)\,C^{\rm R}_{jj}(x-x',\omega) \tag{8}$$

where $C^{\rm R}_{jj}(x-x',\omega)$ denotes the Fourier transform of $C^{\rm R}_{jj}(x-x',t) = -i\theta(t)\,\langle[j(x,t),j(x',0)]\rangle$, and with $E_x(x,\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}E_x(x,t)$.

d) **1 Point**

Quite generally, the Fourier transform of a retarded correlation function can be found by first calculating the related Fourier transform of the imaginary time correlator to Matsubara frequencies, and then performing the analytical continuation $i\omega_n \to \omega + i\eta$ with $\eta \to 0^+$. You can thus find the conductance from the Matsubara frequency expression

$$C_{jj}(x-x',\omega_n) = \int d\tau e^{i\omega_n\tau}\,\left(-\langle T_\tau j(x,\tau)j(x',0)\rangle\right). \tag{9}$$

In addition, it is well-known that for a normal state (non-superconducting) system, the diamagnetic contribution to $\langle j_{\rm tot}\rangle$ cancels with part of the paramagnetic term described by C_{jj} to give a finite total conductance. This is, however, a statement for a wire with a quadratic dispersion - in systems with a purely linear dispersion, the case we use for Luttinger liquid calculations, the diamagnetic term does not even arise in the first place. Use the usual Luttinger liquid definitions and the continuity equation to show that the current is indeed given by $j_{\rm tot} = j = ie\partial_\tau\phi(x,\tau)/\pi$. You can then show that

$$C_{jj}(x-x',\omega_n) = \frac{e^2\omega_n^2}{\pi^2}\,\langle\phi(x,\omega_n)\phi(x',-\omega_n)\rangle. \tag{10}$$

e) **1 Point**

Evaluate the function $C_{jj}(x-x',\omega_n)$ for an infinite wire with effective velocity u and Luttinger liquid parameter K. Show that anaytical contiunation yields

$$C_{jj}(x-x',\omega) = \frac{e^2K(-i\omega)}{2\pi}e^{i\omega|x-x'|/u}.$$

Along the way, you may find it helpful to recall the Gaussian integral

$$\langle u_i(q_1)u_j(q_2)\rangle = \frac{\int\mathcal{D}(u)\,u_i^*(q_1)u_j(q_2)\,e^{-\frac{1}{2}\sum_{q,i,j}A_{ij}(q)u_i^*(q)u_j(q)}}{\int\mathcal{D}(u)\,e^{-\frac{1}{2}\sum_{q,i,j}A_{ij}(q)u_i^*(q)u_j(q)}} = \delta_{q_1,q_2}\,A^{-1}_{ij}(q_1)$$

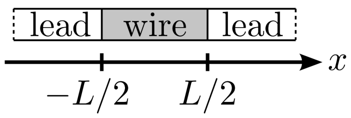
for a vector of fields that satisfies $\mathbf{u}(q) = \mathbf{u}^*(-q)$.

f) **1 Point**

Define the conductance relating the current j and the voltage drop V across the wire as $j = GV$ for the case of an electric field constant in space and time. Compare this result to the finding of exercise sheet 1, where a simple calculation for a single channel of non-interacting electrons gave $G = e^2/h$. What does the observation that all experiments agree with this latter calculation imply for the above Luttinger liquid calculation?

g) **1 Point**

An argument as to why the Luttinger liquid parameter K is not entering the measured conductances was already given by Kane and Fisher in 1992. Namely, they argued that any wire is finite, and that the length of the wire defines an energy scale $\omega_L = v_F/L$, where v_F is the Fermi velocity. What part of the system should thus set the physics at energies (or frequencies) smaller than ω_L, and what does this imply for the conductance? Why can you model a finite size quantum wire attached to higher-dimensional leads as a 1D system of infinite size in which only a central part of size L is interacting, while the outer regions (called the left and right lead) have $K = 1$ (a sketch of this model system can be found on the following picture)?



In this inhomogenous situation, the calculation of $C_{jj}(x - x', \omega_n)$ follows from the differential equation

$$\left(\frac{\omega_n^2}{\pi u(x) K(x)} - \partial_x \frac{u(x)}{\pi K(x)} \partial_x \right) \langle \phi(x, \omega_n) \phi(x', -\omega_n) \rangle = \delta(x - x'), \tag{11}$$

which you may solve independently in the different regions, and then find the full solution by appropriate boundary conditions. What is the value of the conductance in this picture? Hint: you may still assume the electric field to be constant in time, and constant within the wire region, while it vanishes outside the wire region. Note that Eq. (8) then implies a restriction for the coordinate x'. Finally, you may assume that $\langle \phi(x, \omega_n) \phi(x', -\omega_n) \rangle \to 0$ for $x \to \pm\infty$.