

## 2 Classical phase transitions and universality

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### 2.1 Definitions

Order parameter (OP): The OP is an observable  $\varphi$ , for which

$$\langle \varphi \rangle \begin{cases} = 0 & \text{in disordered phase} \\ \neq 0 & \text{in ordered phase} \end{cases}$$

↑  
thermodynamic average ( $T \neq 0$ )  
& quantum expectation value (quantum system)

Remarks:

- $\varphi$  usually local observable:  $\varphi = \varphi(x, t)$

[counter-example: volume enclosed by Fermi surface of a metal]

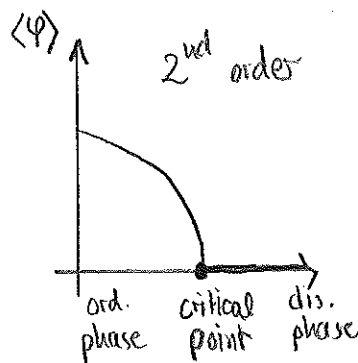
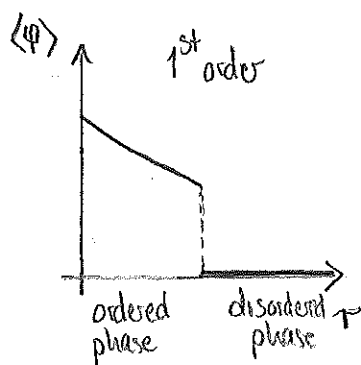
- $\varphi$  not unique

- $\varphi$  sometimes not known [e.g., interaction-driven metal-insulator transition]

Example [Ferromagnet (FM)]:  $\vec{\varphi} = \vec{S}_i$  local magnetization at site  $i$ .

First-order transition: The OP changes discontinuously at the transition.

Continuous transition: The OP varies continuously across the transition.



Critical point: The transition point of a continuous transition.

Correlations: For an order parameter  $\varphi = \varphi(\vec{r}, t)$ , correlation functions can be defined by  $\langle \varphi(\vec{r}, t) \varphi(\vec{r}', t') \rangle$  ("two-point function")

Correlation length  $\xi$ : In stable phase the OP correlation function typically follows an exponential law

$$\langle \varphi(\vec{r}) \varphi(\vec{r}') \rangle - \langle \varphi(\vec{r}) \rangle \langle \varphi(\vec{r}') \rangle \propto e^{-\frac{|\vec{r}-\vec{r}'|}{\xi}}$$

with a correlation length  $\xi$ .

Remarks:

- $\xi$  diverges at a critical point, e.g., for ferromagnetic transition:

$$\langle \varphi(\vec{r}) \varphi(\vec{r}') \rangle - \langle \varphi(\vec{r}) \rangle \langle \varphi(\vec{r}') \rangle \propto \frac{1}{|\vec{r}-\vec{r}'|^{d-2+\eta}}$$

where  $\eta$  is the anomalous dimension.

$\Rightarrow$  correlation function becomes a power law [scale invariance!]

- Near criticality,  $\xi$  is large and becomes the only length scale characterizing the low-energy physics [ $a/\xi \rightarrow 0$ ]

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Spontaneous symmetry breaking: Consider a Hamiltonian  $\hat{H}$  that is

invariant under a symmetry generated by  $\hat{S}$ ;  $[\hat{H}, \hat{S}] = 0$ .

If the system's density operator  $\hat{\rho}$  is not symmetric under  $\hat{S}$ ,

$[\hat{\rho}, \hat{S}] \neq 0$ , then the symmetry  $\hat{S}$  is spontaneously broken.

Remarks:

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- Spontaneous symmetry breaking requires nonanalyticity of thermodynamic potential (otherwise:  $[\hat{H}, \hat{S}] = 0$  would imply  $[e^{-\beta \hat{H}}, \hat{S}] = 0$ )  
 $\Rightarrow$  phase transition

- Singularity can be seen by including a small conjugate field  $h$ :

$$\hat{H} \mapsto \hat{H} - h \Psi$$

Spontaneous symmetry breaking:  $\lim_{h \rightarrow 0^+} \hat{S}(h) \neq \hat{S}(h=0)$

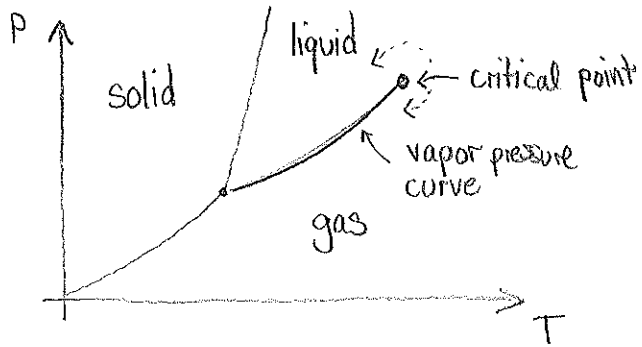
- Spontaneous symmetry breaking impossible in a finite-size system ( $e^{-\beta \hat{H}}$  not singular for  $N < \infty$ )

$\Rightarrow$  spontaneous symmetry breaking means  $\lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} (\dots) \neq \lim_{N \rightarrow \infty} \lim_{h \rightarrow 0} (\dots)$

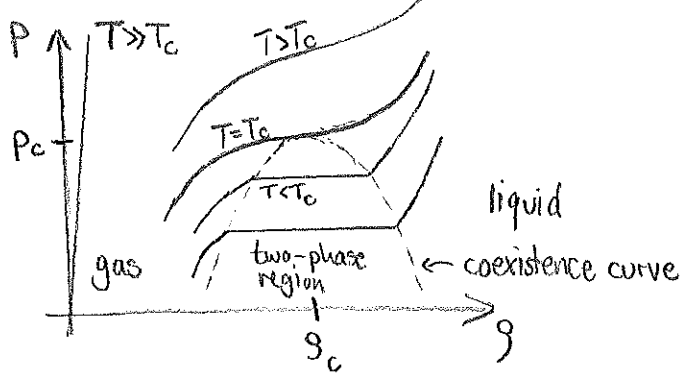
# 2.2 Generic phase diagrams of fluids and magnets

## Fluid

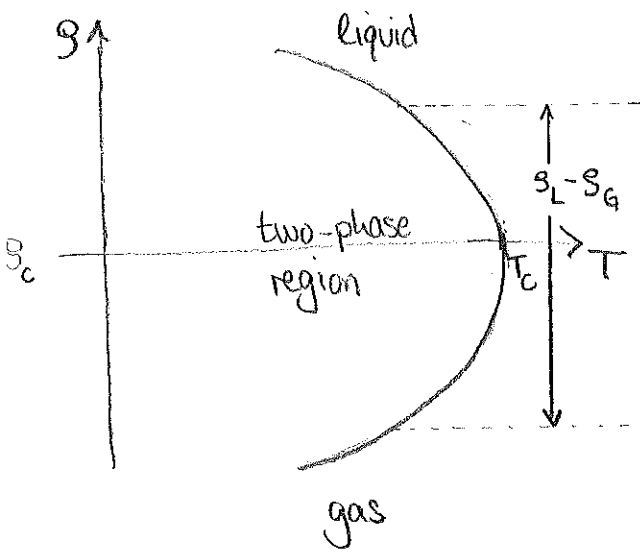
Pressure vs. temperature



pressure vs. density

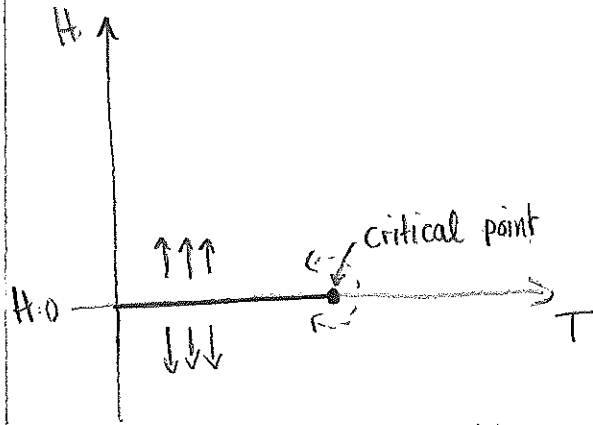


density vs. temperature

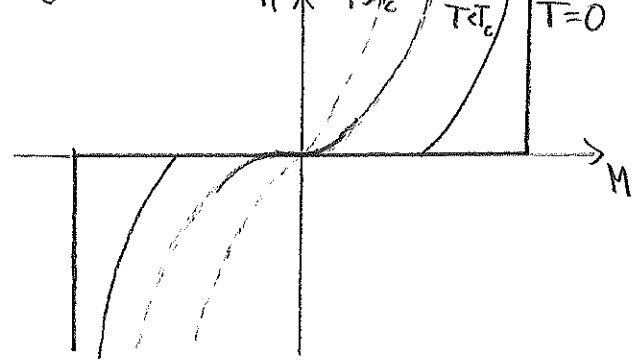


## Magnet

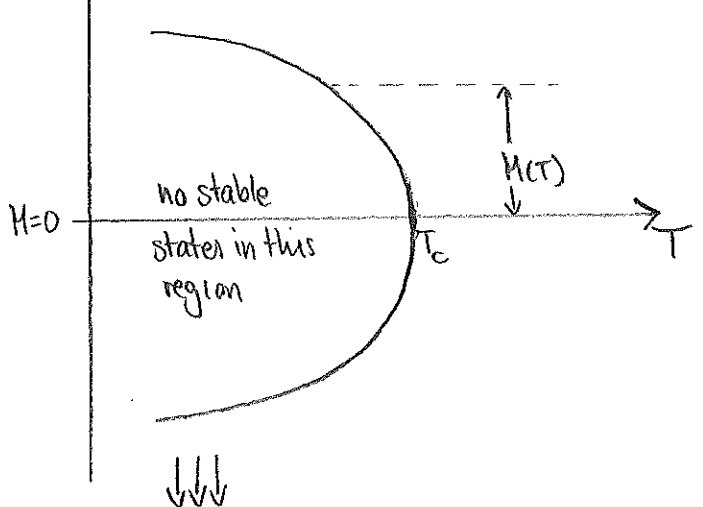
magnetic field vs. temperature



magnetic field vs. magnetization



M vs. T



## 2.3 Landau theory of phase transitions

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Assumption: phase transition uniquely described in terms of a local order parameter  $\varphi$

Idea: Expand a generalized thermodynamic potential  $f$  ("Landau functional") in terms of  $\varphi$   
(conventional potential is singular at transition!)

### Landau functional

Example: (ferromagnet):

Specifying magnetic field  $H$  and temperature  $T$  fixes magnetization  $\varphi \equiv M$ .

Generalized potential:

$$\begin{array}{ccc} f(T, H) & \longmapsto & f(T, H, \varphi) \\ \uparrow \text{thermod. pot.} & & \uparrow \text{Landau potential} \end{array}$$

and assume that  $f(T, H, \varphi)$  is non-singular.

Equilibrium state:

$$\left. \frac{\partial f(T, H, \varphi)}{\partial \varphi} \right|_{\varphi = \varphi_{\text{eq}}} = 0 \quad \text{and} \quad f(T, H) = f(T, H, \varphi_{\text{eq}}(T, H))$$

Expansion of Landau potential (valid near criticality):

$$f(T, H) = f_n + f_0 \left( \frac{a(T)}{2} \varphi^2 + \frac{b(T)}{4} \varphi^4 + \frac{c(T)}{6} \varphi^6 + \dots - \varphi \cdot h \right)$$

where  $h \equiv \frac{H}{f_0}$  is the magnetic field ("conjugate to OP") and

$f_n$  only weakly  $T$  dependent.

Remarks:

- $f$  includes all symmetry-allowed terms, e.g. for  $H=0$ :

[liquid-gas trans.]  
[Ising trans.]

$$\mathbb{Z}_2: \varphi \mapsto -\varphi \quad (\text{"Ising symmetry"})$$

$\Rightarrow$  only even powers in  $\varphi$  allowed

- Vector order parameter  $\vec{\varphi}$ :

[H' beg FM]

$$O(N): \vec{\varphi} \rightarrow R \vec{\varphi}$$

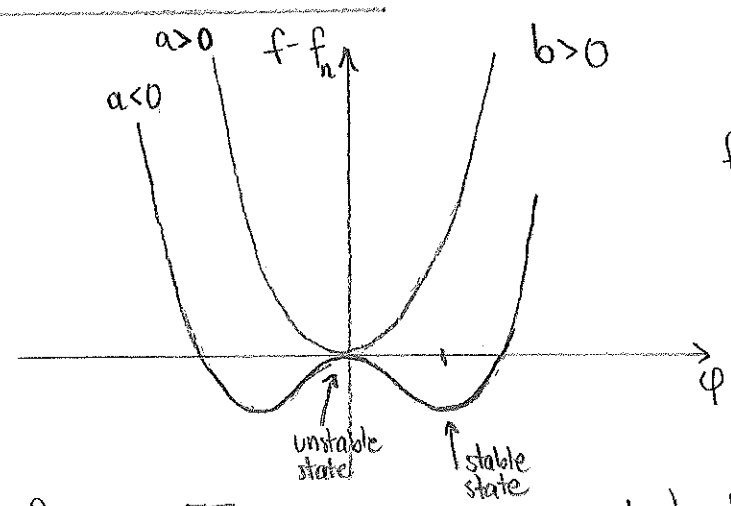
↑  
rotation matrix

$\Rightarrow$  only  $\vec{\varphi}^2 = \vec{\varphi} \cdot \vec{\varphi}$ ,  $(\vec{\varphi}^2)^2$ ,  $(\vec{\varphi}^2)^3$ , ... allowed.

- coefficients  $a(T)$ ,  $b(T)$ , ... are smooth functions of external parameters that preserve the symmetry (in particular, of the temperature)

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Discussion for  $h=0$  and  $b>0$



$$f - f_h = \frac{a}{2} \varphi^2 + \frac{b}{4} \varphi^4$$

If  $b(T) > 0$  for  $T \approx T_c$ , then we can neglect higher-order terms  $\propto O(\varphi^6)$  for  $\varphi$  near  $\varphi_{eq}$ :

$$\left. \frac{\partial f}{\partial \varphi} \right|_{\varphi_{eq}} = 0, \quad \left. \frac{\partial^2 f}{\partial \varphi^2} \right|_{\varphi_{eq}} > 0 \Rightarrow \varphi_{eq} = \begin{cases} 0, & a > 0 \text{ disordered state} \\ \pm \sqrt{\frac{-a}{b}}, & a < 0 \text{ ordered state} \end{cases}$$

Phase transition:

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$$T = T_c \Leftrightarrow a(T) = 0$$

Expansion in reduced temperature  $t = \frac{T - T_c}{T_c}$  (deviation from critical point):

$$a(T) = \alpha t + \mathcal{O}(t^2)$$

where we have assumed that  $a(T)$  is smooth across the transition.

Order parameter:

$$\langle \varphi \rangle(T) = \begin{cases} 0 & , T > T_c \\ \pm \sqrt{\frac{\alpha}{b}(-t)} & , T < T_c \end{cases}$$

Remarks:

• In general  $\langle \varphi \rangle(t < 0) \propto (-t)^\beta$  with a critical exponent  $\beta$   
( $\beta = \frac{1}{2}$  in Landau theory)

• Spontaneously broken  $\mathbb{Z}_2$ :  $\langle \varphi \rangle(t < 0) = \pm \sqrt{\frac{-a}{b}}$

Spontaneously broken  $O(N)$ :  $\langle \vec{\varphi} \rangle(t < 0) = \sqrt{\frac{-a}{b}} \vec{e}_0$

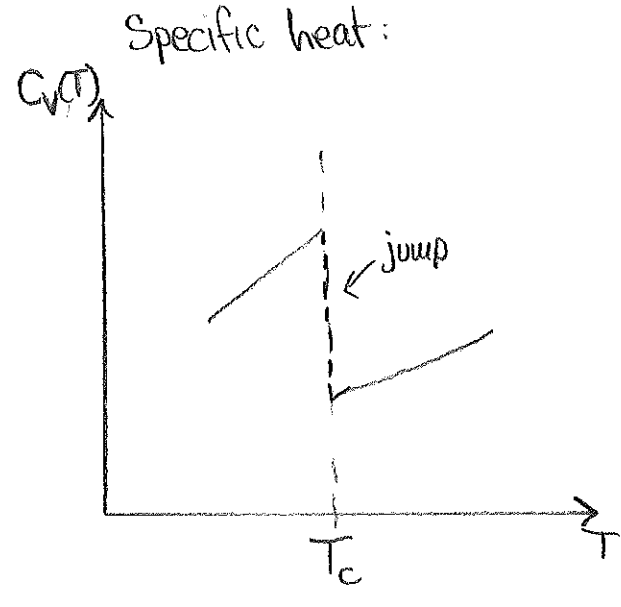
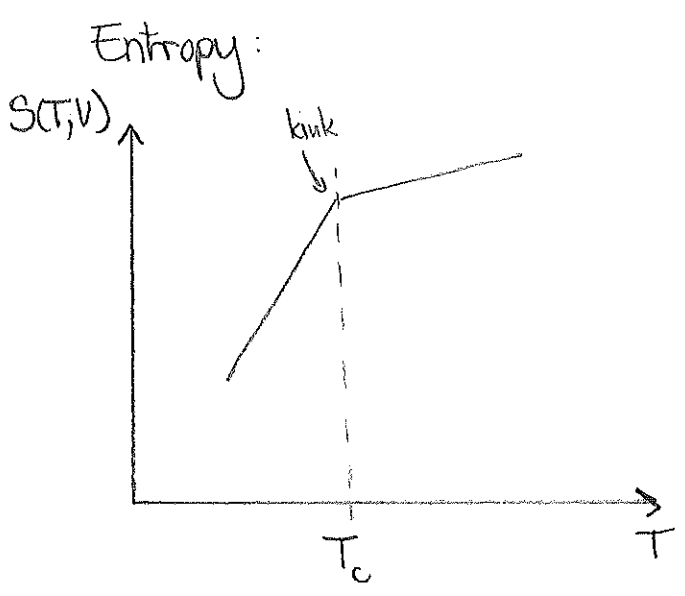
↑ arbitrary direction

Thermodynamic observables:

$$f(T) = f(T, \varphi_{eq}) = \begin{cases} f_n(T) & , T > T_c \\ f_n(T) - f_0 \frac{\alpha^2}{4b} \left(\frac{T_c - T}{T_c}\right)^2 & , T < T_c \end{cases}$$

$$\frac{S}{V} = - \left( \frac{\partial f}{\partial T} \right)_V = \begin{cases} s_0(T) & , T > T_c \\ s_0(T) - f_0 \frac{\alpha^2}{2b} \frac{T_c - T}{T_c} & , T < T_c \end{cases}$$

$$c_v = \frac{1}{V} \left( \frac{\partial S}{\partial T} \right)_V = \begin{cases} c_0 & , T > T_c \\ c_0 + f_0 \frac{\alpha^2}{2b} \frac{1}{T_c} & , T < T_c \end{cases}$$



In general:  $C_V(T) = C_{\pm} |t|^{\alpha} + O(t^2)$   
 with critical exponent  $\alpha$  ( $\alpha=0$  in Landau theory)

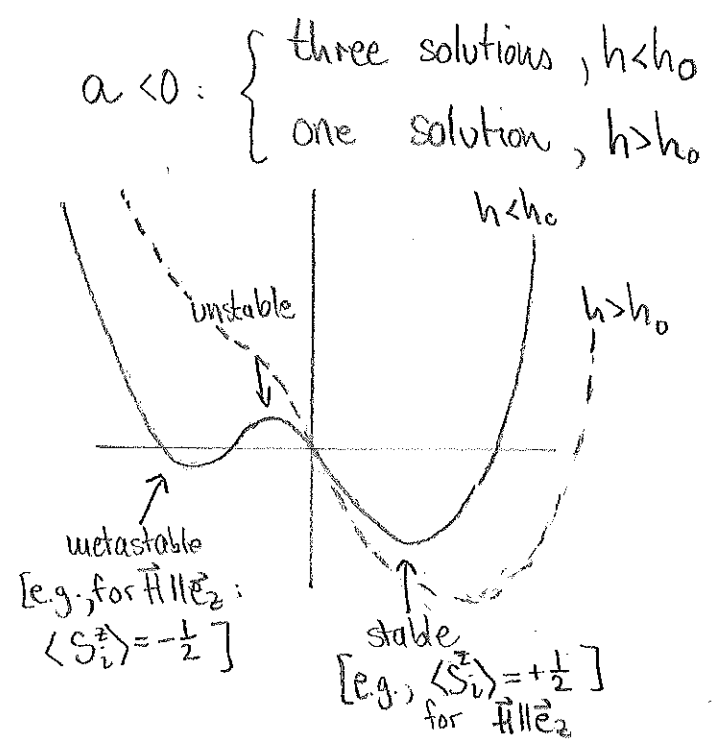
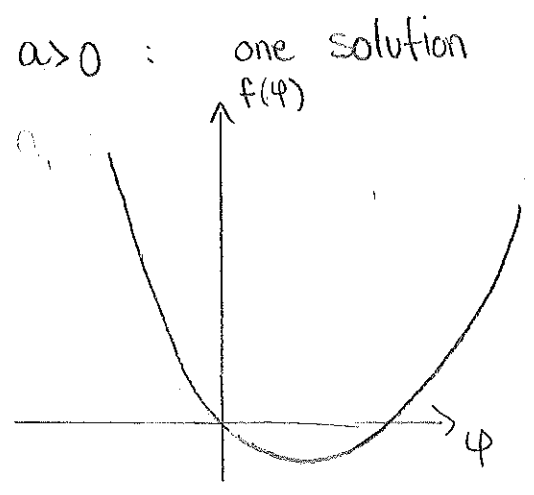
Discussion for  $h=0$  and  $b < 0$  : see exercises

Discussion for  $h \neq 0$  and  $b > 0$

Equilibrium state:

$$\frac{\partial f}{\partial \varphi} = 0 \Rightarrow a\varphi + b\varphi^3 = h \quad \text{with } a = a(T) = \alpha t + O(t^2)$$

(\*)





Susceptibility  $\chi$ :

$$\varphi(h) = \varphi_{\text{spont}} + \chi(T) \cdot h + \mathcal{O}(h^2)$$

with  $\chi := \left. \frac{\partial \varphi}{\partial h} \right|_{h \rightarrow 0}$  the order-parameter susceptibility

and  $\varphi_{\text{spont}}$  the zero-field OP ("spontaneous order")

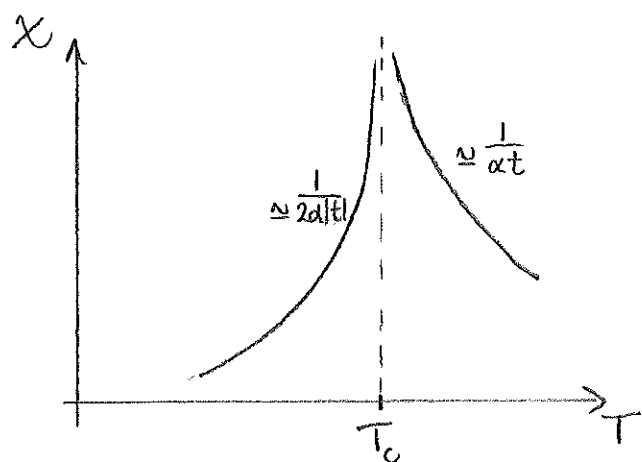
From (\*) for small  $h$ :

$$a(T) [\varphi_{\text{spont}} + \chi(T)h] + b [\varphi_{\text{spont}} + \chi(T)h]^3 = h$$

$$\text{with } \varphi_{\text{spont}} = \begin{cases} 0 & , T > T_c \\ \sqrt{\frac{-a}{b}} & , T < T_c \end{cases}$$

we get by expanding in  $h$  to the leading order:

$$\chi(T) = \begin{cases} \frac{1}{\alpha t} & , T > T_c \\ \frac{1}{2\alpha |t|} & , T < T_c \end{cases} \quad \Rightarrow \quad \lim_{T \rightarrow T_c} \chi(T) = \infty$$



In general:

$$\chi(T) \propto |t|^{-\gamma} + \mathcal{O}(t^2) \quad \text{with critical exponent } \gamma$$

( $\gamma = 1$  in Landau theory)

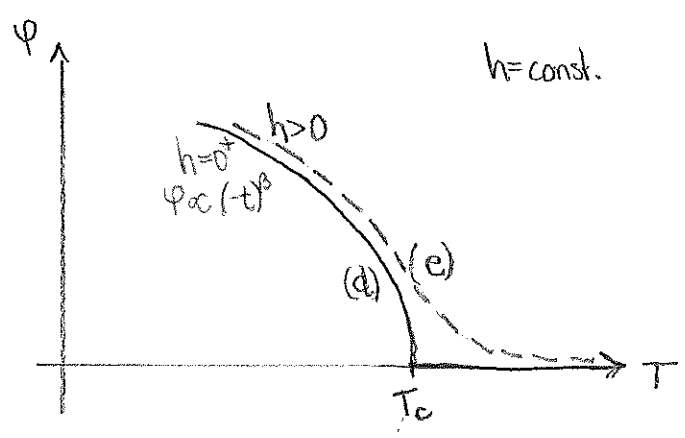
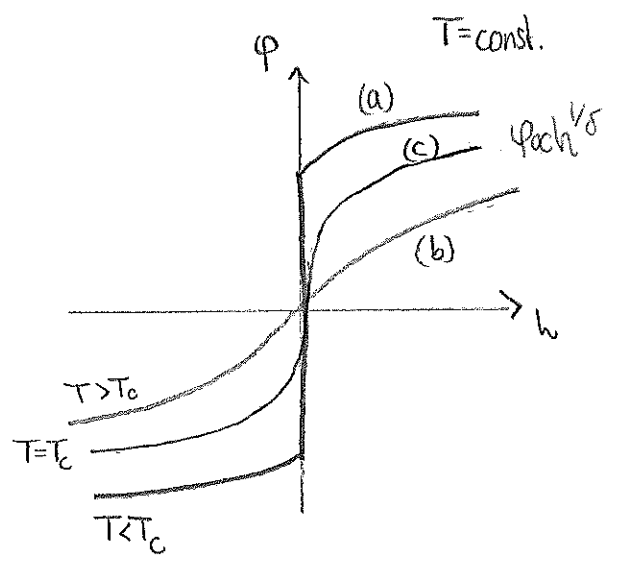
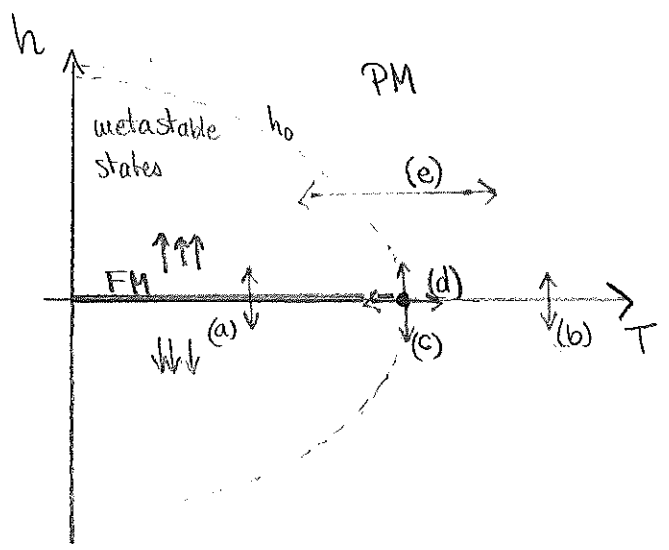
From (\*) for finite  $h$  at  $T=T_c \Leftrightarrow a(T)=0$  :

$$\varphi = \left(\frac{h}{b}\right)^{1/3} \quad \text{"critical isotherm"}$$

$\Rightarrow$

In general :  $\varphi \propto |h|^{1/\delta}$  with critical exponent  $\delta$   
 ( $\delta=3$  in Landau theory)

Phase diagram :



Remarks:

- Line of first-order transition at  $h=0$  for  $T < T_c$  (a)
- Here: the two states at  $h=0^+$  and  $h=0^-$  are related by  $Z_2$  symmetry  
 $\Rightarrow$  latent heat  $Q = \Delta S = 0$  (typically:  $Q \neq 0$ )
- Metastable states for  $h < h_0$  and  $T < T_c$   
 $\Rightarrow$  hysteresis in  $\Psi(h)$  near  $h \approx 0$
- Continuous transition at  $T = T_c$  and  $h = 0$  (c, d)
- "Crossover" for  $T > T_c$  (b) or  $h \neq 0$  (e) (no phase transition)

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Ginzburg-Landau theory and spatial correlations

Allow spatial variations for OP or fluctuations:

$$\varphi \mapsto \varphi(\vec{r})$$

where  $\vec{r}$  is a continuous space coordinate and  $\varphi(\vec{r})$  is a smooth function, capturing variations on lengths scales much larger than microscopic lengths ("continuum-limit description")

[important to compute correlations, but also for inhomogeneous condensates  $\rightarrow$  superconductors in field!]

Ginzburg-Landau functional:

$$f(T, H, \varphi(\vec{r})) = f_n + f_0 \left[ \frac{a}{2} \varphi(\vec{r})^2 + \frac{b}{4} \varphi(\vec{r})^4 + \sum_0^2 (\vec{\nabla}_{\vec{r}} \varphi(\vec{r}))^2 - \varphi(\vec{r}) h \right] + O(\varphi^6, \nabla^4, \nabla^2 \varphi^4)$$

$\uparrow$   
 energy cost of spatial variations

Ginzburg-Landau (free) energy:

$$F = \int d^d \vec{r} f(\vec{r})$$

Correlations :

$$G(\vec{r}, \vec{r}') = \langle \psi(\vec{r}) \psi(\vec{r}') \rangle - \langle \psi(\vec{r}) \rangle \langle \psi(\vec{r}') \rangle = \langle \psi(\vec{r}) \psi(\vec{r}') \rangle - \psi_0^2$$

"propagator"

with  $\psi_0 \equiv \langle \psi(\vec{r}) \rangle$  homogenous OP.

In the large-distance limit for  $T < T_c$ :

$$\lim_{|\vec{r}-\vec{r}'| \rightarrow \infty} \langle \psi(\vec{r}) \psi(\vec{r}') \rangle = \psi_0^2 \neq 0 \quad \text{"long-range order"}$$

and thus  $G(\vec{r}, \vec{r}')$  measures correlations of fluctuations around  $\psi_0$ .

Expansion of  $\psi$  around  $\psi_0$ :

$$\psi(\vec{r}) = \psi_0 + \delta\psi(\vec{r})$$

Free energy:

$$F(\psi(\vec{r})) = F(\psi_0) + \delta F \quad \text{with} \quad \delta F = \frac{f_0}{2} \int d^d \vec{r} \delta\psi(\vec{r}) \left( \alpha t + 3b\psi_0^2 - \sum_0^2 \vec{\nabla}_{\vec{r}}^2 \right) \delta\psi(\vec{r})$$

with  $\alpha(T) = \alpha t + O(t^2)$  and the terms linear in  $\delta\psi$  vanish due to the equilibrium condition  $\frac{\delta F}{\delta\psi} \Big|_{\psi_0} = 0$ .

Fourier decomposition  $\delta\psi(\vec{r}) = \int \frac{d^d k}{(2\pi)^d} \delta\psi(\vec{k}) e^{i\vec{k} \cdot \vec{r}}$  :

$$\delta F = \frac{k_B T}{2} \int \frac{d^d k}{(2\pi)^d} \delta\psi(\vec{k}) G^{-1}(\vec{k}) \delta\psi(\vec{k})$$

with

$$G(\vec{k}) = \langle \delta\psi(\vec{k}) \delta\psi(-\vec{k}) \rangle = \frac{k_B T}{f_0 [A(\vec{k}) + \sum_0^2 \vec{k}^2]}$$

$$\text{and } A(\vec{k}) \equiv \alpha t + 3b\psi_0^2 = \begin{cases} \alpha t & , T > T_c \\ -2\alpha t & , T < T_c \end{cases}$$

Fourier back transform:

$$G(\vec{r}, \vec{r}') = \int \frac{d^d k}{(2\pi)^d} G(\vec{k}) e^{i\vec{k}(\vec{r}-\vec{r}')} \propto \frac{e^{-\frac{r}{\xi(T)}}}{r^{\frac{d-1}{2}}} \xi(T)^{\frac{3-d}{2}}, \quad r = |\vec{r}-\vec{r}'|$$

with correlation-length  $\xi$ :

$$\xi(T) = \begin{cases} \frac{\xi_0}{\sqrt{\alpha t}} & , T > T_c \\ \frac{\xi_0}{\sqrt{2\alpha|t|}} & , T < T_c \end{cases}$$

which diverges for  $T \rightarrow T_c$ .

In general:  $\xi(T) \propto |t|^{-\nu}$  with critical exponent  $\nu$   
( $\nu = \frac{1}{2}$  in Ginzburg-Landau theory)

Remarks:

- Divergent length scale  $\xi$ : locally ordered "islands" of increasing size
- $\varphi(\vec{r})$  varies slowly near criticality  $\Rightarrow$  justifies gradient expansion a posteriori
- Also: divergent correlation time  $\tau_c \rightarrow \infty$  ("critical slowing down")

At criticality: ( $A(T)=0$ ):

$$G(\vec{k}) \propto \frac{1}{k^2} \Rightarrow G(r) \propto r^{2-d} \quad (d > 2)$$

In general:  $G(r) \propto r^{2-d-\eta}$  with "anomalous dimension"  $\eta$ .  
( $\eta = 0$  in Landau-Ginzburg theory)

# Fluctuations and Ginzburg criterion

Landau theory can be "derived" as a saddle-point solution of a field theory formulated as a functional integral (see chapter 3)  
 ⇒ neglects fluctuations ("mean-field approximation")

Ginzburg criterion for the validity of Landau theory:

$$\frac{\langle \delta\psi(\vec{r}=\xi(T)\vec{e}) \delta\psi(0) \rangle}{\varphi_0^2} \propto \frac{\xi(T)^{2-d}}{|t|} \propto |t|^{\frac{d-4}{2}} \xrightarrow{|t| \rightarrow 0} \begin{cases} 0 & d > 4 \\ \infty & d < 4 \end{cases}$$

Annotations:  
 -  $\xi(T) \propto |t|^{-1/2}$   
 -  $\propto \xi^{2-d}$   
 -  $\propto (-t)^{1/2} \quad (T < T_c)$   
 - "effective relative size of fluctuations"

⇒ Landau theory asymptotically exact for  $d > 4$

### Remarks:

- $d_c^+ = 4$  is called "upper critical dimension"
- $d_c^+$  depends on (some general) system properties
- Critical exponents of Landau theory ("mean-field exponents") exact for  $d > d_c^+$
- Logarithmic corrections directly at  $d = d_c^+$
- Landau theory fails at criticality for  $d < d_c^+$
- Mean-field exponents still observable in systems in which the numerical prefactor in the Ginzburg criterion is small (e.g., conventional superconductors:  $t_{cr} \approx 10^{-10} \ll 1$ )
- Consistent account for fluctuations beyond mean field: renormalization group (see chapter 4)

# 2.4 Critical exponents and universality

At criticality:

$\xi \rightarrow \infty \Rightarrow$  fluctuations on all length scales  
"scale invariance"

Observables follow power laws:

$A \sim x^y$  with some critical exponent  $y = \alpha, \beta, \gamma, \delta, \eta, \nu, \dots$

$\uparrow$   $\uparrow$   
 $(\varphi, C_v, \chi, \xi, \dots)$   $(t, h, \dots)$

Common critical exponents:

Observable	Exponent	Definition	Conditions	MF value (GL) $d > d_c$	Example: Ising ( $d=3$ )
Specific heat	$\alpha$	$C \propto  t ^{-\alpha}$	$t \rightarrow 0, h=0$	$\alpha=0$	$\alpha \approx 0.110$
Order parameter	$\beta$	$\rho \propto (t)^{\beta}$	$t \rightarrow 0, h=0$	$\beta = \frac{1}{2}$	$\beta \approx 0.326$
Susceptibility	$\gamma$	$\chi \propto  t ^{-\gamma}$	$t \rightarrow 0, h=0$	$\gamma=1$	$\gamma \approx 1.237$
Critical isotherm	$\delta$	$h \propto  t ^{\delta} \text{sgn}(t)$	$t=0, h \rightarrow 0$	$\delta=3$	$\delta \approx 4.789$
Correlation length	$\nu$	$\xi \propto  t ^{-\nu}$	$t \rightarrow 0, h=0$	$\nu = \frac{1}{2}$	$\nu \approx 0.630$
Correlation function	$\eta$	$G(r) \propto  r ^{-d+2-\eta}$	$t=0, h=0$	$\eta=0$	$\eta \approx 0.0363$
Correlation time	$z$ "anomalous dimension" "dynamical exponent"	$\tau_c \propto \xi^z \propto  t ^{-\nu z}$	$t \rightarrow 0, h=0$	n/a	n/a

Remark:

- $z$  will play a special role in quantum phase transitions, c.f. Chapter 6
- Scale invariance of power laws: E.g.,

$$C \propto |t|^{-\alpha} \propto \xi^{\alpha/\nu} \xrightarrow{\xi \rightarrow b\xi} b^{\alpha/\nu} C$$

# Universality :

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- Critical exponents identical for microscopically completely different
- Systems fall into universality classes : E.g.,  $Z_2$  universality = { Ising model, liquid-gas critical point, ... }
- Universality classes characterized by only very few general properties :  
dimension  $d$ , symmetry of order parameter, presence or absence of long-range interactions
- Phenomenological reason:  $\xi \rightarrow \infty \Rightarrow$  microscopic details become irrelevant
- Deeper understanding: renormalization group (RG), see Chapter 4



## 2.5 Scaling hypothesis

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Assumption:  $\xi$  is only length scale near criticality

Scaling transformation:

$$\begin{aligned}x &\rightarrow b x && \text{lengths} \\t &\rightarrow b^{y_t} t && \text{reduced temperature} \\h &\rightarrow b^{y_h} h && \text{external field}\end{aligned}$$

Scale invariance: change in length can be compensated by change in  $t$  and  $h$  (for appropriate  $y_t, y_h$ )

Scaling hypothesis for free energy density:

$$f_s(t, h) = b^{-d} f_s(b^{y_t} t, b^{y_h} h)$$

↑ singular part of free energy

$f_s$  is homogenous in both  $t$  and  $h$ .

Consequence:

All critical exponents can be related to  $y_t$  and  $y_h$  and there exist thus only two independent exponents in static critical phenomena.

Example (correlation length):

$$\text{Scaling transf.: } \xi \propto |t|^{-\nu} \xrightarrow{t \rightarrow b^{y_t} t} b^{-y_t \nu} |t|^{-\nu} \propto b^{-y_t \nu} \xi \xrightarrow{x \rightarrow b^x} b^{1-y_t \nu} \xi$$

$$\text{Scale invariance: } \xi \mapsto \xi \Rightarrow \underline{\underline{\nu = \frac{1}{y_t}}}$$

Example (free energy density at  $h=0$ ):

$$f_s(t, h=0) = b^{-d} f_s(b^{y_t} t, 0)$$

$$\text{Choose } b \text{ such that } b^{y_t} |t| = 1 : b^{-d} = |t|^{\frac{d}{y_t}} = |t|^{d\nu}$$

$$f_s(t, h=0) = |t|^{d\nu} f_s(\pm 1, 0) \Rightarrow \underline{\underline{f_s(t, h=0) \propto |t|^{d\nu}}}$$

Example (specific heat):

$$C_v \propto T \frac{\partial^2 f}{\partial T^2} \propto |t|^{d\nu-2} \Rightarrow \underline{\underline{\alpha = 2 - d\nu}} \quad \text{"Josephson's law"}$$

Scaling hypothesis for correlation function ( $h=0$ ):

$$G(r; t, h=0) = \frac{f_G\left(\frac{r}{\xi(t)}\right)}{r^{d-2+\eta}}$$

Critical point ( $t=0$ ):

$$G(r; t=0, h=0) = \frac{f_G(0)}{r^{d-2+\eta}}$$

Susceptibility:

$$\begin{aligned} \chi(\vec{q} \rightarrow 0) &= \int d^d r G(r; t) = \int dr f_G(r |t|^\nu) r^{1-\eta} \quad \left[ \begin{array}{l} x = r |t|^\nu \\ dx = |t|^\nu dr \end{array} \right] \\ &= \underbrace{\int dx f_G(x) x^{1-\eta}}_{\text{const.}} |t|^{-(2-\eta)\nu} \end{aligned}$$

$$\Rightarrow \underline{\underline{\gamma = (2-\eta)\nu}} \quad \text{"Fisher's law"}$$

Other scaling relations ( $h > 0$ ):

$$\alpha + 2\beta + \gamma = 2 \quad \text{"Rushbrooke's law"}$$

$$\alpha + \beta(\delta + 1) = 2 \quad \text{"Griffiths' law"}$$

Remarks:

- Scaling hypothesis is an assumption that can be justified with RG (chapter 4)
- It holds for  $d < d_c^+$ , but not for  $d > d_c^+$  ("hyperscaling violation")
- Mean-field exponents violate Josephson's law (unless  $d = d_c^+$ )

27.4.18