Exercises for "Quantum Phase Transitions" SS 18

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1. Interchange of limits in the classical Ising chain (4 points)

Consider a classical Ising chain with M sites and nearest-neighbor exchange ($K > 0$):

$$H = -\sum_i K\sigma_i\sigma_{i+1}. \tag{1}$$

(a) Write down the partition function

$$Z \equiv \sum_{\{\sigma_i=\pm 1\}} e^{-H} \tag{2}$$

in transfer-matrix representation.

(b) Evaluate Z as well as the spin-spin correlation function $\langle \sigma_i\sigma_0\rangle$ exactly.

(c) Investigate now two possible routes to obtain the correlation function in the limit of large K (small T).

(i) Approximate $\langle \sigma_i\sigma_0\rangle$ first for large K and then take the limit $M \to \infty$.

(ii) Take first the limit $M \to \infty$ and then approximate $\langle \sigma_i\sigma_0\rangle$ for large K.

(d) Why are the results different? Explain which different physical situations the two routes correspond to.

Hint: Determine the energy Δ required to create a domain wall between a region with all spins up and a region with all spins down, and think in terms of domain walls.

2. Classical Ising chain with next-nearest neighbor interactions (3 points)

Consider the infinite ($M \to \infty$) classical Ising chain with first and second neighbor exchange ($K_1, K_2 > 0$)

$$H = -\sum_i \left(K_1\sigma_i\sigma_{i+1} + K_2\sigma_i\sigma_{i+2}\right). \tag{3}$$

(a) Write down the partition function Z as a transfer-matrix product.

Hint: Think of the model in terms of "superspins" with 4 states, each of which represents a block of two neighboring Ising spins.

(b) Is it possible to diagonalize the transfer matrices using a unitary transformation?

(c) Determine the energy Δ required to create a domain wall.

(d) Determine the correlation length ξ of the model for large K_1, K_2, and show that $\xi = (a/2)e^{\Delta}$ (a is the lattice spacing).

Hint: Recall the lesson learned from Problem 1.

3. Relation between energy gap and correlation length (3 points)

We wish to show now that the relationship $\xi = (a/2)e^{\Delta}$ holds quite generally, i.e., independently of the model and, to some extent, temperature. To this end, we think of the spin configurations in terms of domain walls and assume the domain walls to be statistically uncorrelated from each other (i.e., we neglect possible interactions between the domain walls).

(a) Argue that the density of domain walls is given by $\rho = (1/a)e^{-\Delta}$. Consequently, it is sufficient to show $\xi = 1/(2\rho)$.

(b) Consider a long chain of length $Ma \gg \xi$ with $N = \rho Ma$ domain walls. The probability that any given domain wall is between 0 and $x > 0$ is $q = x/(Ma)$. Use the statistical independence of the domain walls to argue that

$$\langle \sigma(x)\sigma(0)\rangle = \sum_{j=0}^{N} (-1)^j q^j (1-q)^{N-j} \frac{N!}{j!(N-j)!}. \tag{4}$$

(c) Evaluate the above expression in the limit $N, M \to \infty$, while $\rho = N/(Ma)$ is finite, to show the desired result.