6. Quantum spin liquids

Quantum spin liquid = State of local magnetic moments at temperatures $T \ll$ exchange interaction J without magnetic long-range order, without spontaneous symmetry breaking, with emergent gauge field & topol. order, with fractionalized excitations

„quantum paramagnet"

In contrast to classical spin liquids (which have highly degenerate ground states), QSL have only small (topological) ground-state degeneracies.

6.1. Valence bonds: Solids, resonating liquids, and trivial states

Assume SU(2)-symmetric (= Heisenberg) Hamiltonian

$$H = \sum J_{ij} \vec{S}_i \cdot \vec{S}_j$$

for spins 1/2 on regular lattice

Paramagnetic states can be constructed from pairwise singlets (building block)

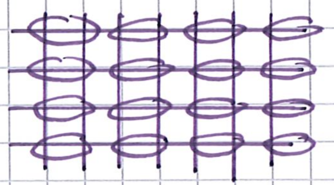
$$(\cdot_i \;\; \cdot_j) = |ij\rangle = 1/\sqrt{2} \left(|\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle \right)$$

Now construct many-body states, where every spin 1/2 has one partner for singlet formation. (→ total spin $S=0$)

$$|VB\rangle = \prod_{\substack{\text{pairs} \\ \{ij\}}} |ij\rangle$$

(a) Solid : Singlet arrangement breaks lattice symmetry
$\hat{=}$ One preferred singlet configuration

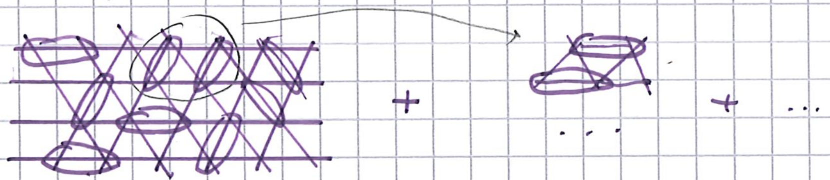
Chain  translation broken (realized in $J_1 - J_2$ model)

Square lattice  translation and rotation broken (realized in $J_1 - J_2$ and $J - Q$ models)

„valence bond solids" – these are NOT spin liquids

States have discrete degeneracies due to broken lattice symmetries.

(b) Resonating liquid : Lattice symmetries preserved
$\hat{=}$ Superposition of many singlet configurations

Triangular lattice  (likely realized in $J_1 - J_2$ model)

$$|\psi\rangle = \sum_{\substack{\text{all dimer} \\ \text{coverings } \alpha}} \lambda_\alpha \underbrace{\prod_{\{ij\}_\alpha} |ij\rangle}_{\substack{\text{valence bond} \\ \text{state } \alpha}}$$

Superposition of all dimer coverings restores lattice symmetries. Processes  can be understood as resonances ⟶ „resonating valence bond" (RVB) state (Anderson 1973)

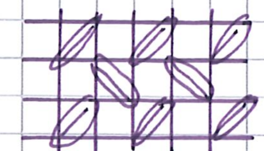
$\hat{=}$ Quantum spin liquid

($\exists$ variants with broken time reversal, partially broken lattice sym. etc.)

(c) Trivial : Singlet pattern reflects lattice symmetry
$\hat{=}$ one preferred singlet configuration

Ladder 

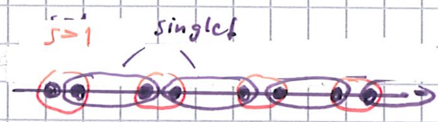
Shastry – Sutherland lattice



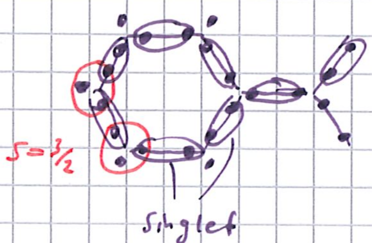
Such states can also occur for $S > \frac{1}{2}$,

can be understood by splitting S into $S = \frac{1}{2}$ pieces.

$S = 1$ chain
(Affleck – Kennedy – Lieb – Tasaki)
AKLT

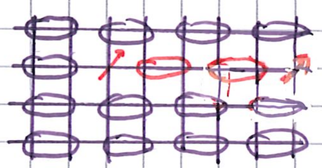


$S = \frac{3}{2}$ honeycomb lattice

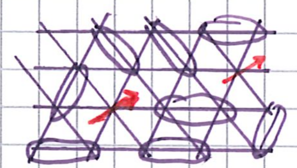


Remarks:

- Different valence-bond states are not orthogonal ( , ).
- RVB states may involve only short-range (eg. nearest-neighbor) pairs or longer-range pairs as well.
- Even for short-range RVB states not all correlation function decay exponentially.
- Trivial VB states and VB solids have conventional (unfractionalized $S=1$) excitations, while RVB states typically have fractionalized $S = \frac{1}{2}$ excitations.



confinement of $S = \frac{1}{2}$ spinons due to misplaced dimers in VBS state



no confinement in RVB state
$\rightarrow$ $S = \frac{1}{2}$ spinons asymptotically free

- Complete description of RVB liquids involves emergent gauge fields

6.2. Parton mean-field theory

(historically RVB mean-field theory)

Often, quantum spin liquids and their descendants can be described by simple mean-field theories. To this end, one employs one of the possible spin representations, and approximates the Hamiltonian such that constituent particles (typically bosons or fermions ⟶ partons) behave like free (!) particles.

Example: Heisenberg model for $S = 1/2$ spins

Fermionic representation $\vec{S}_i = \frac{1}{2} \sum'_{\alpha\beta} f^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} f_{i\beta}$, $\alpha, \beta = \uparrow, \downarrow$

Interaction
$$\vec{S}_i \cdot \vec{S}_j = -\frac{1}{2} \sum_{\alpha\beta} f^\dagger_{i\alpha} f_{j\alpha} f^\dagger_{j\beta} f_{i\beta} + \frac{1}{4}$$

Now decouple quartic interaction, with goal to describe SU(2)-invariant state (spin liquid $\hat{=}$ no symmetry breaking).

Two possible decouplings in spin-singlet channel:

(i) $\langle \sum'_\alpha f^\dagger_{i\alpha} f_{j\alpha} \rangle \sum_\beta f^\dagger_{j\beta} f_{i\beta}$ „particle-hole"

(ii) $\langle f^\dagger_{i\uparrow} f^\dagger_{j\downarrow} - f^\dagger_{i\downarrow} f^\dagger_{j\uparrow} \rangle (f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow})$ „particle-particle"

This defines two different fermionic mean-field theories.

Case (i): particle-hole decoupling

$$H_{MF} = -\frac{J}{2} \sum_{\langle ij\rangle \alpha} \left[\left(f^\dagger_{i\alpha} f_{j\alpha} \chi_{ji} + \text{h.c.} \right) - |\chi_{ij}|^2 \right] + \sum'_{i\alpha} \lambda_i \left(f^\dagger_{i\alpha} f_{i\alpha} - 1 \right)$$

↑

Lagrange multiplier for Hilbert-space constraint

Mean-field parameters χ_{ij} and λ_i determined from

$$\chi_{ij} = \sum_\alpha \langle f_{i\alpha}^\dagger f_{j\alpha} \rangle$$

$$1 = \sum_\alpha \langle f_{i\alpha}^\dagger f_{i\alpha} \rangle$$

H_{MF} is a free-fermion Hamiltonian and can be solved exactly. Self-consistency must be achieved numerically.

Qualitative results:

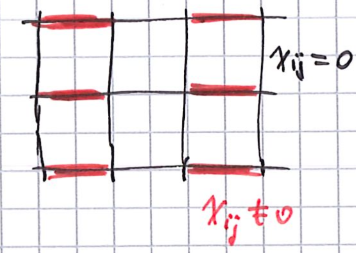
- high T: only solution is $\lambda_i = 0$, $\chi_{ij} = 0$
  $\curvearrowright$ f fermions are dispersionless at zero energy
  $\hat{=}$ decoupled free spins (e.g. $\chi \propto 1/T$ Curie)

- low T: $\exists$ solutions with $\chi_{ij} \neq 0$.
  Simplest case $\chi_{ij} = \chi \neq 0$ uniform
  $\curvearrowright$ f fermions hop on original lattice,
  i.e. they form a band (of "spinons"),
  with a Fermi surface (since band is half-filled)
  $\hat{=}$ Spin liquid with spinon fermi surface

<u>Remarks</u>

1) Decoupling in particle-particle channel leads to spinon pairing, i.e. mean-field Hamiltonian has BCS form.
Spectrum then does not have Fermi surface, but full gap (or nodal points).
This phase is <u>not</u> a superconductor (but a spin liquid), as the $f_{i\alpha}$ do <u>not</u> carry electric charge.

2) Often solutions with non-uniform χ_{ij} exist.
($\hat{=}$ breaking of lattice translation symmetry)
Extreme case: only some $\chi_{ij} \neq 0$, others $= 0$.



This represents a mean-field description of a valence-bond solid!

3) Solutions may have complex χ_{ij}. This is akin to particles hopping in the presence of a magnetic field ($t_{ij} \longrightarrow t_{ij}\, e^{ie/\hbar\, (\vec{A}_j - \vec{A}_i)\cdot \hat{n}_{ij}}$), i.e. the spinons move in the presence of a (fictitious) self-generated flux.
$\hat{=}$ so-called „flux phase"
If the flux through every plaquette is equal, translation invariance is preserved. If this flux corresponds to $\frac{1}{2}\Phi_0$ (Φ_0 = flux quantum); i.e. hopping phase around plaquette is π, then time reversal is also preserved.

4) The $f_{i\alpha}$ carry spin-½ by construction. For SU(2) symmetry, S is a good quantum number
$\rightarrow$ the $f_{i\alpha}$ represent spin-½ spinons.
Given that $\vec{S} = \frac{1}{2} \sum_{\alpha\beta} f_\alpha^+ \vec{\sigma}_{\alpha\beta} f_\beta$, spins fractionalize into two fermionic spinons.

If SU(2) symmetry is broken, fractionalization can still occur, but the partons do not carry well-defined spin (see e.g. Kitaev model).

5) The picture of non-interacting spinons only holds at mean-field level. Beyond that, both short-range interactions and the coupling to a gauge field occur. (see later)
(The latter may even destabilize the phase under consideration!)

6) Mean-field theory is in general <u>uncontrolled</u> and, to some degree, arbitrary, as both the spin representation and the decoupling are NOT unique.
One can, however, find certain large-N limits ($SU(2) \to SU(N)$ with specific representation) where particular mean-field theories become <u>exact</u> (!) in the limit $N \to \infty$. Corrections to mean-field can then be calculated in a $1/N$ expansion.
Practically, one often chooses a mean-field theory suitable to describe the phase of interest, and then uses it to calculate physical properties.

7) Mean-field theories often yield thermal phase transitions (high T: $\chi = 0$, low T: $\chi \neq 0$) in cases where there is no physical phase transition.
Such transitions ($\hat{=}$ artefacts of mean-field theory) disappear once the coupling to the gauge field is included. Unphysical transitions also imply unphysical order parameters (here: χ); these do then not correspond to observables.

8) Bosonic mean-field theories (e.g. using Schwinger-boson representation) often display Bose-Einstein condensation at $T = 0$ (and possibly also at low $T > 0$), corresponding to magnetic long-range order.

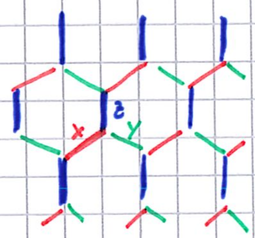
6.3. Kitaev honeycomb model (Kitaev 2006)

(one of very few exactly soluble spin models in $d=2$)
(yield $\mathbb{Z}_2$ quantum spin liquid with Majorana-fermion excitations)

$$\hat{H} = -J_x \sum_{\langle ij\rangle_x} \sigma_i^x \sigma_j^x - J_y \sum_{\langle ij\rangle_y} \sigma_i^y \sigma_j^y - J_z \sum_{\langle ij\rangle_z} \sigma_i^z \sigma_j^z$$

with bond-dependent Ising interactions (non-commuting!)

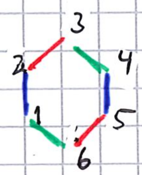
on honeycomb lattice; $\sigma \hat{=}$ Pauli matrix ($\curvearrowright$ spins $1/2$)



Strong anisotropy in spin space
$\hat{=}$ strong spin-orbit coupling

a) Conserved quantities

For each plaquette p define



$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

site 1 has y and z interactions along plaquette loop
$\curvearrowright$ choose σ_1^x here (the component not involved)

One can show $[\hat{H}, \hat{W}_p] = 0 \quad \forall p$.

(can be defined for any closed loop!)

Eigenvalues of W_p (product of Pauli!): $W_p = \pm 1$

Eigenstates of $\hat{H}$ can be chosen as eigenstates of $\hat{W}_p$!

W_p is called ($\mathbb{Z}_2$) flux. $\{W_p\}$ defines flux sector.

b) Majorana representation

Majorana fermion : fermion which is its own antiparticle

$$c_j^2 = 1 \,, \qquad c_i c_j = -c_j c_i \quad \text{if } i \neq j$$

Can be constructed from canonical fermion : $c_{2k} = a_k + a_k^\dagger$, $c_{2k+1} = \frac{1}{i}(a_k - a_k^\dagger)$

Two Majoranas $\hat{=}$ one canonical fermion $\left(a_k = \frac{1}{2}(c_{2k} + i c_{2k+1}) \right)$

$\curvearrowright$ Hilbert space of single Majorana fermion has dimension $\sqrt{2}$ (!)

(Or: Majorana fermions can only occur in pairs)

Spin $\frac{1}{2}$ representation via Majorana fermions:

$$\underline{\sigma^\alpha = i b^\alpha c} \,, \quad \alpha = x, y, z \quad \curvearrowright \quad 4 \text{ Majoranas } b^{x,y,z}, c$$

Pauli

Constraint $b^x b^y b^z c = 1$

Kitaev interaction in Majorana representation:

$$\sigma_i^\gamma \sigma_j^\gamma = (i b_i^\gamma c_i)(i b_j^\gamma c_j) = -i \underbrace{(i b_i^\gamma b_j^\gamma)}_{u_{ij}} c_i c_j$$

Define $\hat{u}_{ij} = i b_i^\gamma b_j^\gamma$ ($\gamma = x, y,$ or z) on $\underline{\gamma}$-bond

One can show $[\hat{H}, \hat{u}_{ij}] = 0 \quad \forall \langle ij \rangle$

Eigenvalues of $\hat{u}_{ij}$ are $u_{ij} = \pm 1$.

Resulting Hamiltonian (in sector with fixed u_{ij}):

$$\boxed{H_u = \frac{i}{4} \sum_{\langle ij \rangle_\gamma} A_{ij} c_i c_j} \quad \text{with} \quad \underline{A_{ij} = 2 J_\gamma u_{ij}} = \text{const}$$

$\langle ij \rangle$ is γ bond

This is a free-fermion hopping problem (!).

The sign degree of freedom in $u_{ij} = \pm 1$ corresponds to a static Z_2 gauge field.

(Recall: U(1) gauge field $\curvearrowright$ hopping acquires complex phase)

The u_{ij} are related to the conserved fluxes

$$W_p = u_{21} u_{23} u_{43} u_{45} u_{65} u_{61} \qquad (\text{ATTN: } u_{ij} = -u_{ji})$$

$c \hat{=}$ „matter fermions"
$b \hat{=}$ „gauge fermions"

c) Solution of Majorana hopping problem (in given flux sector!)

Re-write $H_u = \frac{i}{2} \left(c_A^T \; c_B^T \right) \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} \begin{pmatrix} c_A \\ c_B \end{pmatrix}$

N-components

with $c_A = \begin{pmatrix} c_{1,A} \\ \vdots \end{pmatrix}$, $c_B = \begin{pmatrix} c_{1,B} \\ \vdots \end{pmatrix}$ vector of matter Majorana operators for the A & B sublattices, and hopping matrix $\underset{N \times N}{M_{ij}} = J_\gamma u_{ij}$.

H_u can be diagonalized via singular-value decomposition of M; for general $\{u_{ij}\}$ this has to be done numerically.

$M = U S V^T$, S diagonal with $S_{ii} = \varepsilon_i \geq 0$ („eigenvalues")

Define $b_A^T = c_A^T U$, $b_B^T = c_B^T V$

$$\curvearrowright \quad H_u = i \sum_{m=1}^{N} \varepsilon_m b_{m,A} b_{m,B}$$

Combine into canonical fermions $a_m = \frac{1}{2} (b_{m,A} + i b_{m,B})$.

$$\curvearrowright \quad H_u = \sum_{m=1}^{N} \varepsilon_m (2 a_m^+ a_m - 1)$$

Ground state of H_u is vacuum of a_m (since $\varepsilon_m \geq 0$).

Excitation energies ε_m depend of gauge-field configuration $\{u_{ij}\}$ (more precisely, they depend on the flux configuration $\{W_p\}$).

d) Spin-liquid ground state

Global lowest-energy state is in flux-free sector, $W_p = +1 \;\forall p$ (can be proven analytically). Then, matter spectrum ε_m can be obtained by Fourier transformation. Result for $J_x = J_y = J_z \equiv J$:

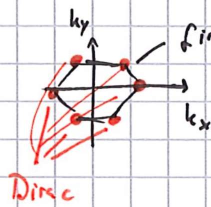
$$\varepsilon_k = |J| \sqrt{3 + f_k} \quad , \quad f_k = 2\cos\sqrt{3}k_y + 4\cos\tfrac{3}{2}k_x \cos\tfrac{\sqrt{3}}{2}k_y$$

real space

nearest-neighbor distance $\equiv 1$

This is positive-energy part of graphene spectrum (!), with Dirac points at $K, K' = (\pm\frac{4}{3}\pi, 0)$.

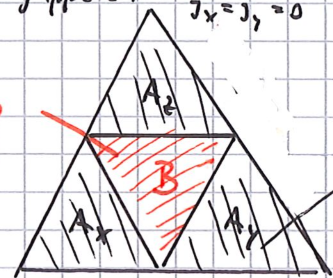
reciprocal space



For $|J_x| > |J_y| + |J_z|$

or $|J_y| > |J_x| + |J_z|$

or $|J_z| > |J_x| + |J_y|$ the ε_k spectrum is gapped.



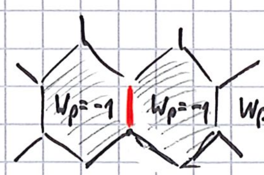
All phases are spin liquids!
(no broken symmetries)

Two types of excitations:

- matter fermions, with spectrum ε_k (gapless in phase B)
- Z_2 fluxes (i.e. plaquettes with $W_p = -1$)

  These are gapped. For $J_x = J_y = J_z \equiv J$ the flux gap is $\Delta_f \approx 0.26|J|$.

Note: Flipping <u>one</u> u_{ij} $(+1 \to -1)$ creates two Z_2 fluxes.



Note: Sign of J irrelevant for solution (!)

e) Physical properties of Kitaev spin liquid

- **Static spin correlations**

$\langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle$, since $\sigma_i^\alpha = i b_i^\alpha c_i$ flips u_{ij} on α bond,

creating two fluxes,

$\langle \sigma_i^\alpha \sigma_j^\beta \rangle$ is only finite $<$ on-site / for nearest-neighbor i, j (!)

$\hat{=}$ ultra-short-ranged spin correlations

- **Dynamic spin correlations**

$S_{ij}^{\alpha\beta}(t) = \langle \sigma_i^\alpha(t)\, \sigma_j^\beta(0) \rangle$, spatial structure as above

(fluxes conserved under time evolution!)

time / frequency dependence can be calculated analytically

Result for Fourier-transformed $S(q=0, \omega)$:

No sharp mode, but

excitation continuum!

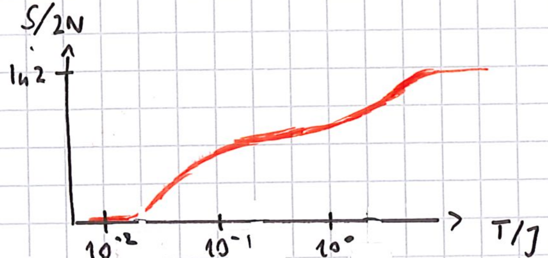
Spin flip decays / dissociates

into Ising flux pair and matter Majorana!

$\boxed{\text{Continuum is generic signature of fractionalization!}}$

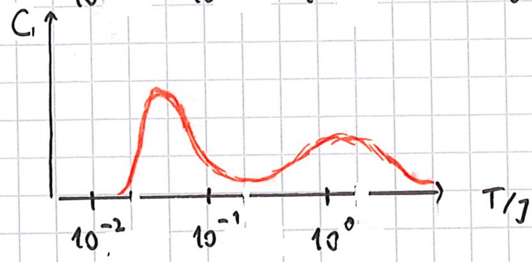
- **Entropy / specific heat**

can be calculated numerically



No thermal phase transition

(spin liquid, no symm. breaking),



two distinct crossovers:

high T: neighboring spins align

low T: fluxes freeze to $W_p = +1$

- **Magnetic field**

Add $H_h = -\sum_i \left(h_x \sigma_i^x + h_y \sigma_i^y + h_z \sigma_i^z \right)$

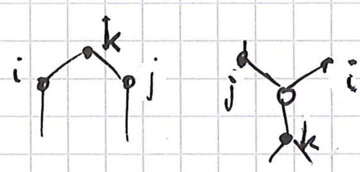
Perturbation theory for small h ↝ project on flux-free sector.

First order: $H_{eff}^{(1)} = 0$ (H_h creates flux pair)

Second order: $H_{eff}^{(2)} \neq 0$ (but only renormalizes hopping)

Third order:

$$H_{eff}^{(3)} \propto \frac{h_x h_y h_z}{\Delta_f^2} \sum_{ijk}' \underbrace{\sigma_i^x \sigma_j^y \sigma_k^z}_{= -i \, (\underbrace{i b_k^x b_k^y b_k^z c_k}_{1}) \, u_{ik} u_{jk} c_i c_j}$$



$H_{eff}^{(3)}$ generates (imaginary) second-neighbor hopping terms.

time-reversal symmetry is broken by h!

Such terms open a gap in phase B, $\Delta_h \propto \frac{h_x h_y h_z}{\Delta_f^2}$,

and yield a band structure with non-zero Chern number

(Haldane 1988)

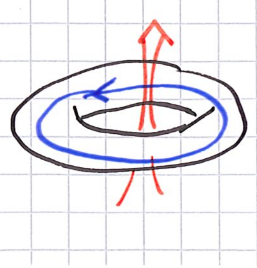
↝ Chiral edge modes exist,

Fluxes behave as non-abelian anyons.

- **Ground-state degeneracy** (general property of Z_2 spin liquid)

On a torus (i.e. with periodic boundary conditions in both directions), there are $\underline{4}$ states which become degenerate ground states in thermodynamic limit. They differ by presence or absence of Z_2 flux through holes of torus.

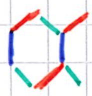
↝ Topological degeneracy!



f) Kitaev model on other lattices

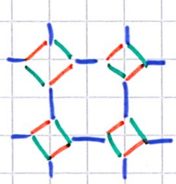
Exactly solvable Kitaev models can be constructed on many tricoordinated lattices. Matter Majorana spectrum depends on lattice

d = 2 honeycomb



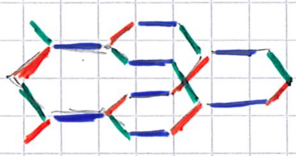
Dirac point

square - octagon



Fermi surface

d = 3 hyper honeycomb



Fermi line

hyper octagon Fermi surface

⋮

For coordination number $\neq 3$ Kitaev-like models are no longer exactly solvable!

g) Heisenberg - Kitaev model

In experimental realizations of Kitaev interactions, other couplings will inevitably be present.

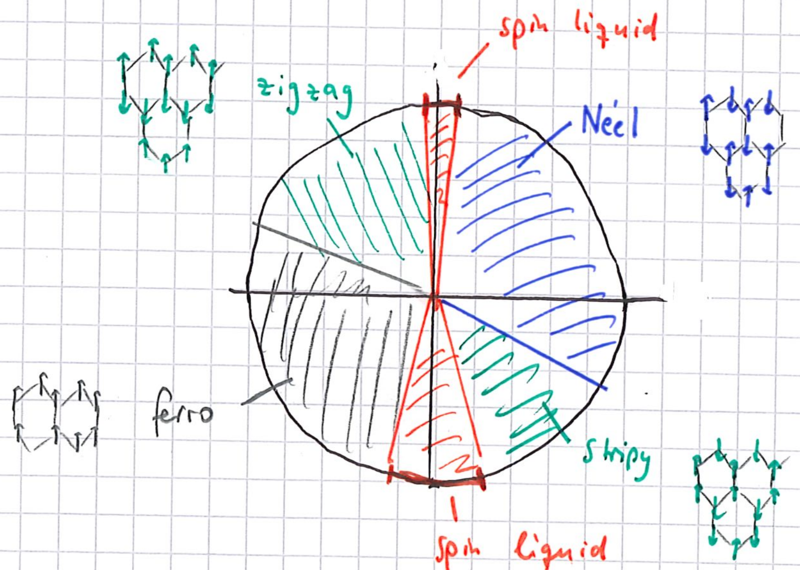
Simplest model: Heisenberg - Kitaev

$$H = J \sum_{\langle ij \rangle}' \vec{S}_i \cdot \vec{S}_j + K \sum_{\langle ij \rangle_\gamma} S_i^\gamma S_j^\gamma$$

Parameterize $J = A \cos\varphi$, $K = 2A \sin\varphi$.

Phase diagram at $T=0$ from numerics:

Spin-liquid phases only cover „small" region of parameter space; all other phases display magnetic order!



Further symmetry-allowed nearest-neighbor terms:

$$H_\Gamma = \Gamma \sum_{\langle ij \rangle_z} \left(S_i^x S_j^y + S_i^y S_j^x \right) + \text{cyclic perm. } (x,y,z)$$

$$H_{\Gamma'} = \Gamma' \sum_{\langle ij \rangle_z} \left(S_i^x S_j^z + S_i^z S_j^x + S_i^y S_j^z + S_i^z S_j^y \right) + \text{cyclic p.}$$

6.4. Partons and emergent gauge fields

Quantum spin liquids typically come with fractionalization and an internal gauge structure, i.e., the low-energy theory involves fractionalized particles (partons) coupled to an emergent gauge field.

Why? Hilbert space spanned by partons is larger than that of original degrees of freedom (spins)

$\rightsquigarrow$ Redundant description, redundancy expressed via gauge transf.

Different fractionalization schemes $<$ different partons; different gauge symmetry (Z_2, $U(1)$, $SU(2)$, ...)

Example 1

Kitaev model $\longrightarrow$ Z_2 gauge theory

$$S^\alpha = \frac{1}{2} i\, b^\alpha c \quad , \quad u_{ij} = b_i^\alpha b_j^\alpha$$

Local Z_2 transformation
$$c_i \to -c_i$$
$$b_i^\alpha \to -b_i^\alpha$$
$$u_{ij} \to -u_{ij} \quad \forall j$$

leaves all S^α invariant
leaves spectrum of Hamiltonian invariant

Kitaev model is special because Z_2 gauge field is static.
Small perturbations added to Kitaev lead to $[\hat{H}, \hat{u}_{ij}] \neq 0$

$\rightsquigarrow$ gauge field fluctuates.

Elementary gauge-field excitation is single plaquette with flux, $W_p = -1$.
This is a Z_2 vortex (particle acquires minus sign when travelling around), typically called vison. Beyond pure Kitaev model visons acquire dispersion!

Example 2

Abrikosov fermion construction (spinon metal) $\longrightarrow$ U(1) gauge field

$$\vec{S} = \frac{1}{2} f_\alpha^+ \vec{\sigma}_{\alpha\beta} f_\beta \quad (\alpha,\beta = \uparrow,\downarrow) \qquad \text{with} \quad \sum_\alpha f_\alpha^+ f_\alpha = 1$$

$\uparrow$ Pauli

Parton mean-field theory involves

$$\chi_{ij} = \sum_\alpha \langle f_{i\alpha}^+ f_{j\alpha} \rangle = |\chi_{ij}| e^{i A_{ij}}$$

phase

In path-integral language, constrained can be enforced by Lagrange multiplier $\lambda_i (\sum_\alpha \bar{f}_{i\alpha} f_{i\alpha})$.

Local U(1) transformation

$$f_i \longrightarrow e^{i\phi_i} f_i$$
$$\lambda_i \longrightarrow \lambda_i + \partial_\tau \phi_i$$
$$A_{ij} \longrightarrow A_{ij} + (\phi_j - \phi_i)$$

from action term $\bar{f} \partial_\tau f$

leaves S and action invariant.

A and λ take role of „space" and „time" component of U(1) gauge field.

As in QED, elementary excitation of gauge field is photon, with dispersion $\omega = c|k|$.

Alternative construction of gauge-field theories for spin liquids:

Start from nearly ordered magnetic state and parameterize its fluctuations.

1) Collinear fluctuations, with order parameter field $\vec{\phi}(\vec{r})$

$$\vec{S}_i = \text{Re}\left(\vec{\phi}\, e^{i\vec{Q}\cdot\vec{r}_i}\right) = \vec{n}_1 \cos(\vec{Q}\cdot\vec{r}_i) + \vec{n}_2 \sin(\vec{Q}\cdot\vec{r}_i),$$

Ordering wavevector $\quad \vec{n}_1 \parallel \vec{n}_2 \;\hat{=}\;$ collinear

Parameterize $\vec{\phi}$ in terms of „spinons":

$$\vec{\phi} = \frac{1}{2}\sum_{\alpha\beta} z_\alpha^+ \vec{\sigma}_{\alpha\beta} z_\beta \,, \qquad \sum_\alpha |z_\alpha|^2 = 1$$

Invariant under U(1) transformation $\quad z_\alpha \to z_\alpha e^{i\phi}$

2) Non-collinear fluctuations

$$\vec{S}_i = \text{Re}\left(\vec{\phi}\, e^{i\vec{Q}\cdot\vec{r}_i}\right) = \vec{n}_1 \cos(\vec{Q}\cdot\vec{r}_i) + \vec{n}_2 \sin(\vec{Q}\cdot\vec{r}_i)$$

$$\vec{n}_1 \cdot \vec{n}_2 = 0 \;\hat{=}\; \text{non-collinear}$$

Parameterize

$$\vec{\phi} = \vec{n}_1 + i\vec{n}_2 = \frac{1}{2}\sum_{\alpha\beta\gamma} \varepsilon_{\alpha\beta}\, z_\beta\, \sigma_{\beta\gamma}\, z_\gamma$$

Invariant under Z_2 transformations $\quad z_\alpha \to \eta z_\alpha \quad$ with $\eta = \pm 1$

Continuum-limit theory in case 1) takes form (CP^1 model)

U(1) gauge field

$$S = \int d^dx\, d\tau \Big[|(\partial_r - iA_r) z_\alpha|^2 + |(\partial_\tau - iA_\tau) z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{4e^2}(\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \Big]$$

Physically, gauge field describes non-magnetic (singlet) excitations.

In this U(1) theory, condensing z_α yields magnetic (Néel) state. „Higgs"

Alternatively, BCS pairing of z_α breaks U(1) symmetry down to Z_2

$\curvearrowright$ Z_2 spin liquid.

6.5. Experimental signatures of quantum spin liquids

- No (dipole ≡ spin) order,
  No magnetic Bragg peaks, No internal fields
- Either no thermal phase transition (2d $\mathbb{Z}_2$, 3d $U(1)$ spin liquid)
  or 3d Ising transition (3d $\mathbb{Z}_2$ spin liquid)
  proliferation of vortex loops
- Separated from asymptotic high-field phase by quantum phase transition (both phases are non-ordered, but QSL is topological)
- Continua in dynamic spin response (instead of sharp modes).
  ATTN: exceptions possible!
- Excitations may be fully gapped (some $\mathbb{Z}_2$ SL)
  or gapless (some $\mathbb{Z}_2$ SL, all $U(1)$ SL),
  visible in specific heat and thermal transport

  $\mathbb{Z}_2$ SL with spinon Fermi surface: $C/T \propto$ const
  $U(1)$ SL (with photon excitation, in 3d): $C/T \propto \ln T$

  Gapless SL may behave like thermal metal (i.e. thermal conductivity $\propto T$ at low T), but is of course electric insulator.

6.6. Candidate models and materials (essentially all $S=\frac{1}{2}$)

- Honeycomb Kitaev (or Heisenberg - Kitaev) — $\alpha-RuCl_3$ (only SL in small field window?)

- Kagome-lattice Heisenberg — Herbertsmithide $ZnCu_3(OH)_6Cl_2$

- Triangular-lattice J_1-J_2 Heisenberg — ?

- Triangular-lattice Hubbard (weak Mott) — κ-(BEDT)TTF organics

- Square-lattice J_1-J_2 Heisenberg — ?

- Pyrochlore quantum spin ice — $Pr_2Hf_2O_7$, $Yb_2Ti_2O_7$ (order at very low T)