Consider the Kagome lattice shown below which forms a two-dimensional hexagonal net with a three-atomic unit cell (“kagome” is the Japanese word for a traditional basket with such a pattern).

Compute the bandstructure for a nearest-neighbor tight-binding model for spinless fermions on the Kagome lattice.

**Guideline:** The Hamiltonian can be written as \( H_K = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + \text{h.c.}) \). The primitive lattice vectors are given by \( a_1 = a(1,0) \) and \( a_2 = a(1/2, \sqrt{3}/2) \). The basis atoms are located at \( r_1 = 0 \) (A/red), \( r_2 = a_1/2 \) (B/green) and \( r_3 = a_2/2 \) (C/blue). The unit cell is indicated by the yellow rhomboid. In the following, each unit cell will be labeled with \( x \) and \( y \) coordinates \((n,m)\), see above. Instead of \( c_i^\dagger c_j \) we are using the operators \( A_{(n,m)}^\dagger, B_{(n,m)}^\dagger, C_{(n,m)}^\dagger \), which annihilate (create) an electron on the corresponding sublattice \( A, B, C \) within the \((n,m)\)-th unit cell. Thus we can write all nearest-neighbor hopping processes (see purple arrows above plus hermitian conjugation) as

\[
H_K = -t \sum_{n,m} \left[ C_{n,m}^\dagger A_{n,m} + B_{n,m}^\dagger C_{n,m} + A_{n,m}^\dagger B_{n,m} + C_{n,m-1}^\dagger A_{n,m} + B_{n-1,m+1}^\dagger C_{n,m} + A_{n+1,m}^\dagger B_{n,m} + \text{h.c.} \right]
\]

The Fourier transform of these terms can be conveniently arranged into a 3x3 matrix, the Bloch matrix, and its eigenvalues are the energy bands. Calculate and sketch these bands in the Brillouin zone.

**Hint:** Determine the characteristic equation of the 3x3 matrix and use the identity

\[
\cos^2\left(\frac{k \cdot a_1}{2}\right) + \cos^2\left(\frac{k \cdot a_2}{2}\right) + \cos^2\left(\frac{k \cdot (a_1 - a_2)}{2}\right) = 2 \cos\left(\frac{k \cdot a_1}{2}\right) \cos\left(\frac{k \cdot a_2}{2}\right) \cos\left(\frac{k \cdot (a_1 - a_2)}{2}\right) + 1.
\]
2. Strong-Coupling Limit of the Hubbard Model

Consider a real nearest-neighbor hopping on an arbitrary lattice,

\[ H_0 = \sum_{\langle ij \rangle} h_{ij} \quad \text{with} \quad h_{ij} = -t \sum_{\sigma=\uparrow,\downarrow} \left( c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.} \right) \] (2)

If we want to describe the effect of Coulomb repulsion between the electrons, this can be accomplished in the simplest way using the Hubbard interaction,

\[ H_I = U \sum_i n_{i\uparrow} n_{i\downarrow}, \] (3)

where \( n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \) is the number operator. At half filling, i.e., one particle per lattice site, and for strong \( U \) hopping processes of the electrons are suppressed. This is the strong-coupling limit where only the spin degree of freedom of the electrons remains at low energies.

a) 5 Points

Show that, for very large \( U \) at half filling, \( H = H_0 + H_I \) corresponds to the low-energy spin Hamiltonian

\[ H_{\text{spin}} = J \sum_{\langle ij \rangle} \left[ \frac{1}{2} \left( S_i^+ S_j^- + S_i^- S_j^+ \right) + S_i^z S_j^z - \frac{1}{4} \right] \] (4)

with \( J = (4t^2)/U \). This is the isotropic Heisenberg model.

Guideline: Note that formally one has to perform second-order perturbation theory in \( 1/U \), i.e., \( H_{\text{spin}} = -T U^{-1} T^\dagger \) where \( T \) and \( U \) are matrices. Consider two sites for which the low-energy states \( |s\rangle \) at half filling are singly occupied, \( |s\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \). Virtual hopping processes result in states \( |d\rangle \in \{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \) where one site is empty and the other doubly occupied. Matrix elements of \( T \) are given by \( \langle s| h_{12} |d\rangle \) and of \( U \) by \( \langle d| H_I |d\rangle \). In general, we expect in second-order perturbation theory an effective spin Hamiltonian with matrix elements \( \langle s| J_{\mu\nu} S_\mu^\dagger S_\nu |s'\rangle \) which can be compared with the matrix elements of \( H_{\text{spin}} \). The constant \( c \) sets the energy zero point. The use of ladder operators \( S_\mu^\pm \) might be helpful.

b) 3 Points

How does the resulting spin Hamiltonian change when imaginary, spin-dependent hopping is considered instead of (2)? As a concrete example, consider the spin-orbit type term on the honeycomb lattice

\[ H'_0 = i\lambda \sum_{\langle ij \rangle, \mu} \nu_{ij} c_{i\alpha}^\dagger \sigma_{\alpha\beta}^\mu c_{j\beta}. \] (5)

Nearest-neighbor bonds \( \langle ij \rangle_\mu \) differ now by the involved Pauli matrices \( \sigma^\mu \) as indicated in the figure below; \( \nu_{ij} = \pm 1 \) depending on whether hopping is clockwise (\( \nu_{ij} = +1 \)) or counter-clockwise (\( \nu_{ij} = -1 \)).

Perform the calculation for one of the three different bonds (blue, red, or green) and guess the solution of the others.

c) 3 Points

Now repeat the calculation for a Hubbard model with both kinetic terms, i.e., \( \tilde{H} = H_0 + H'_0 + H_I \).