Technische Universität Dresden
Institut für Theoretische Physik

Prof. M. Vojta
M. Nayga

Theory of Frustrated Magnetism
Problem set 5

Summer term 2020

Parton Mean-Field Theory **18 Points**

The parton mean-field treatment of a Heisenberg Hamiltonian

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \tag{1}$$

allows for a description of non-magnetic states or spin liquids. This is accomplished by introducing Abrikosov fermions or pseudo-fermion operators $f_{i\alpha}^{(\dagger)}$ ($\alpha =\uparrow, \downarrow$),

$$\vec{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{i\beta} \ , \tag{2}$$

and enforcing a constraint on the Hilbert space, $f_{i\alpha}^\dagger f_{i\alpha} = 1$, which forbids unphysical states. Summation over repeated indices is always implied.

Rewrite the Hamiltonian (1) in terms of pseudo-fermions and show that it becomes

$$H = -\frac{1}{2} J \sum_{\langle ij \rangle} f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta} + J \sum_{\langle ij \rangle} \left(\frac{1}{2} n_i - \frac{1}{4} n_i n_j \right) \ . \tag{3}$$

Hint: Use $\vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\alpha'\beta'} = 2\delta_{\alpha\beta'}\delta_{\alpha'\beta} - \delta_{\alpha\beta}\delta_{\alpha'\beta'}$.

The second term in (3) is a constant which we neglect in the following. Note that the Hamiltonian (3) only corresponds to the original spin Hamiltonian (1) in the subspace of one fermion per site (which is ensured by the constraint). The usual strategy of the parton mean-field theory consists of two steps: (i) Replace the constraint by its average, $\langle f_{i\alpha}^\dagger f_{i\alpha} \rangle = 1$, which is enforced by a site-dependent and time-independent Lagrange multiplier $\lambda_i (f_{i\alpha}^\dagger f_{i\alpha} - 1)$ in the Hamiltonian. (ii) Replace two of the operators in the first term of (3), $f_{i\alpha}^\dagger f_{j\alpha}$, by its expectation value χ_{ij}. Show that the resulting mean-field Hamiltonian reads

$$H = -\frac{1}{2} J \sum_{\langle ij \rangle} \left(\left[f_{i\alpha}^\dagger f_{j\alpha} \chi_{ji} + \text{h.c.} \right] - |\chi_{ij}|^2 \right) + \sum_i \lambda_i \left(f_{i\alpha}^\dagger f_{i\alpha} - 1 \right) \tag{4}$$

Derive the self-consistency equation for χ_{ij}; alternatively, you can find χ_{ij} by minimizing the mean-field energy. In the following, we consider three specific trial states.

(1) Dimer state

A simple ansatz χ_{ij} for a dimer state on the square lattice is given by $\lambda_i = 0$ and

$$\chi_{i,i+\hat{\boldsymbol{x}}} = \frac{\chi_1}{2}\left(1 + (-1)^{i_x}\right) \qquad \text{and} \qquad \chi_{ij} = 0 \quad \text{else.} \tag{5}$$

Show that the mean-field ground-state corresponding to this ansatz satisfies the self-consistency equation $\chi_{ij} = \langle f^\dagger_{i\alpha} f_{j\alpha}\rangle$ and the constraint $\langle f^\dagger_{i\alpha} f_{i\alpha}\rangle = 1$. Compute and sketch the mean-field excitation spectrum and determine the value of χ_1 at zero temperature.

(2) π-flux state

A more sophisticated ansatz on the square lattice is given by $\lambda_i = 0$ and

$$\chi_{i,i+\hat{\boldsymbol{x}}} = i\chi_2 \qquad \text{and} \qquad \chi_{i,i+\hat{\boldsymbol{y}}} = i\chi_2(-1)^{i_x} \tag{6}$$

where χ_2 is a real constant. Show that the corresponding ansatz satisfies the self-consistency equation and the constraint. Compute and sketch the mean-field excitation spectrum and determine the value of χ_2 at zero temperature. Do not try to solve the final momentum integrals.

(3) Homogeneous spin liquid state

Eventually we consider a homogeneous ansatz on the square lattice which is given by $\lambda_i = 0$ and

$$\chi_{i,i+\hat{\boldsymbol{x}}} = \chi_3 \qquad \text{and} \qquad \chi_{i,i+\hat{\boldsymbol{y}}} = \chi_3 \tag{7}$$

where χ_3 is a real constant. Show that the corresponding ansatz satisfies the self-consistency equation and the constraint. Compute and sketch the mean-field excitation spectrum and determine the value of χ_3 at zero temperature.