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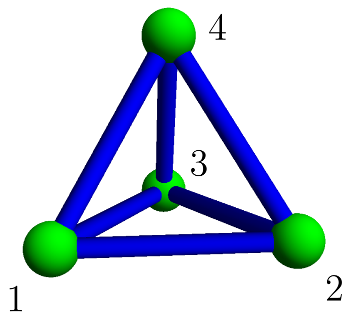
Theory of Frustrated Magnetism
Problem set 2

Summer term 2023

1. Four spins 1/2 on a tetrahedron — 9 Points

Consider a tetrahedron with a spin $S = 1/2$ on each corner (the green balls).



a) **3 Points**

In the following, we denote the two-dimensional spin 1/2 representation as $\boldsymbol{\frac{1}{2}}$. Find the total content of irreducible spin representations, *i.e.*, compute the tensor product $\boldsymbol{\frac{1}{2}} \otimes \boldsymbol{\frac{1}{2}} \otimes \boldsymbol{\frac{1}{2}} \otimes \boldsymbol{\frac{1}{2}}$. Use the formula

$$\boldsymbol{S} \otimes \boldsymbol{S'} = |\boldsymbol{S} - \boldsymbol{S'}| \oplus (|\boldsymbol{S} - \boldsymbol{S'}| + \boldsymbol{1}) \oplus \ldots \oplus (\boldsymbol{S} + \boldsymbol{S'} - \boldsymbol{1}) \oplus (\boldsymbol{S} + \boldsymbol{S'}) \tag{1}$$

which reproduces for $\boldsymbol{S} = \boldsymbol{S'} = \boldsymbol{\frac{1}{2}}$ the well-known $\boldsymbol{\frac{1}{2}} \otimes \boldsymbol{\frac{1}{2}} = \boldsymbol{0} \oplus \boldsymbol{1}$: the irreducible representations of two coupled spins 1/2 are the one-dimensional singlet representation $\boldsymbol{0}$ and the three-dimensional triplet representation $\boldsymbol{1}$. Note that the spin representations obey the distributive law, $(\boldsymbol{a} \oplus \boldsymbol{b}) \otimes \boldsymbol{c} = (\boldsymbol{a} \otimes \boldsymbol{c}) \oplus (\boldsymbol{b} \otimes \boldsymbol{c})$.

What is the total Hilbert-space dimension of the four coupled spins 1/2 and how many $S^z = 0$ subspaces do they contain?

Hint: Two coupled spins 1/2 decompose into singlet and triplet, one- and three-dimensional representations, adding up to a total dimension of dim = 4. The spin singlet contains only the $S^z = 0$ subspace, the triplet, however, contain subspaces with $S^z = 0, \pm 1$. We see that two coupled spins 1/2 contain in total two subspaces with $S^z = 0$.

b) **4 Points**

Consider the spin 1/2 Heisenberg Hamiltonian $\mathcal{H} = \sum_{i<j} \vec{S}_i \cdot \vec{S}_j$ where $i, j = 1, 2, 3, 4$ correspond to the four corners of the blue tetrahedron. We consider the S^z basis states $|S_1^z, S_2^z, S_3^z, S_4^z\rangle$ with $S_i^z = \uparrow, \downarrow$. In the following, we are interested in the subspace with $S_{\mathrm{tot}}^z = \sum_{i=1}^4 S_i^z = 0$: Calculate the Hamiltonian matrix in this subspace explicitly and diagonalize it. What is the groundstate energy? What is the spin of the ground state? Specify a groundstate eigenvector.

Hint: In order to diagonalize the resulting Hamiltonian matrix feel free to use a computer program of your choice (examples are mathematica, maple, matlab etc.).

c) **2 Points**

Now we want to check if the energies calculated in part b) are correct. Express the Hamiltonian $\mathcal{H}$ through the square of the total spin of the tetrahedron, $\boldsymbol{S}_{\mathrm{tot}}^2$. Then use the results of part a) to find the energies.

2. Linear Spin Wave Theory in the high-field limit **9 Points**

Consider the nearest-neighbor Heisenberg model on a two-dimensional lattice with antiferromagnetic exchange coupling $J > 0$ subject to a strong uniform magnetic field $h > 0$,

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - h \sum_i S_i^z \ . \tag{2}$$

In this limit, the system is fully polarized and the magnon excitations are gapped. If we reduce the magnetic field strength, the magnon gap decreases and vanishes eventually ("magnons become soft modes") which indicates that the polarized phase becomes unstable. We want to calculate the critical magnetic field and the wavevector at which the polarized phase breaks down for various two-dimensional lattices. To do so, we apply linear spin wave theory and introduce Holstein–Primakoff bosons defined as

$$S_i^z = S - a_i^\dagger a_i, \quad S_i^+ = \sqrt{2S - a_i^\dagger a_i}\ a_i, \quad S_i^- = a_i^\dagger \sqrt{2S - a_i^\dagger a_i}\ . \tag{3}$$

Expand the square roots in Eq. (3) for small $n_i \equiv a_i^\dagger a_i \ll S$ to lowest order. Plug these simplified expressions into the Heisenberg Hamiltonian (2) and neglect quartic terms, *i.e.*, four-operator terms.

a) **3 Points**

Calculate the critical field h_c where the minimum of the magnon dispersion becomes zero for the square lattice. Determine at which point $\vec{Q}$ in the Brillouin zone the magnon touches zero energy first.

b) **2 Points**

Repeat the previous calculation for the triangular lattice.

c) **4 Points**

Now repeat the previous calculations for the honeycomb lattice.