Technische Universität Dresden
Institut für Theoretische Physik

Prof. M. Vojta
Dr. B. Frank

Theory of Frustrated Magnetism
Problem set 3

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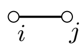
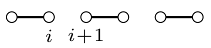
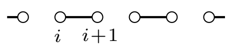
1. Non-Magnetic States I: Majumdar–Ghosh **6 Points**

An obivous way to construct non-magnetic states is by binding spins pairwise into singlets,

$$\frac{1}{\sqrt{2}}\left|\uparrow_i\downarrow_j - \downarrow_i\uparrow_j\right\rangle \equiv \left|\right.$$  $$\left.\right\rangle . \qquad (1)$$

With such *singlet bonds* any lattice can be easily covered, thus resulting in a state with total $S = 0$. *A priori* it is not clear, however, if such a singlet state is the groundstate of a local Hamiltonian.

In one spatial dimension, we consider a spin $S = 1/2$ chain of length N = even (we impose periodic boundary conditions, *i.e.*, lattice site $i + N \equiv i$) and the two dimer states

$$\left|\psi^{\text{even}}\right\rangle = \left|\right.$$  $$\left.\right\rangle \quad \text{and} \quad \left|\psi^{\text{odd}}\right\rangle = \left|\right.$$  $$\left.\right\rangle . \qquad (2)$$

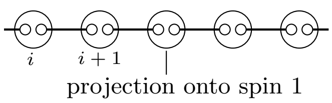
Find a Hamiltonian containing Heisenberg-type spin exchange for which $\left|\psi^{\text{even}}\right\rangle$ and $\left|\psi^{\text{odd}}\right\rangle$ are the groundstates. What is the corresponding groundstate energy?

Hint: Compare the content of irreducible spin representations for three spins 1/2 and for three consecutive sites of the states $\left|\psi^{\text{even}}\right\rangle$ and $\left|\psi^{\text{odd}}\right\rangle$. Use the resulting insight to construct an operator which annihilates three neighboring sites of $\left|\psi^{\text{even}}\right\rangle$ and $\left|\psi^{\text{odd}}\right\rangle$.

2. Non-Magnetic States II: AKLT **6 Points**

The Affleck–Kennedy–Lieb–Tasaki (AKLT) chain is a paradigm for a spin-1 chain, as it is known to behave qualitatively similar to the spin-1 Heisenberg chain. [F. D. M. Haldane conjectured in 1983 that integer-spin Heisenberg chains are fundamentally different from half-odd integer spin chains because their excitation spectrum is gapped. Partly because of this conjecture, Haldane was awarded the Nobel prize in physics in 2016.]

The idea of AKLT is the following: each spin 1 on a given site can be thought of as two "virtual" spins 1/2 which are in a symmetrically state (remember, that $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$, where the singlet $\mathbf{0}$ is a totally antisymmetric and the triplet $\mathbf{1}$ a totally symmetric representation). Virtual spins 1/2 on neighboring sites can now be antisymmetrically coupled into singlet bonds. We can visualize this as

$$|\text{AKLT}\rangle = \left|\right.$$  $$\left.\right\rangle , \qquad (3)$$

where the large circles are lattice sites and the small dots are the virtual spins 1/2. Note that $|\text{AKLT}\rangle$ is translationally invariant, while the $|\psi\rangle$ states from Problem **1** are not.

a) **3 Points**

Construct a Hamiltonian for which $|\text{AKLT}\rangle$ is the groundstate.

Hint: Compare again the content of irreducible spin representations of two adjacent sites of $|\text{AKLT}\rangle$ with that of two coupled spin-1 representations and find an operator which annihilates two neighboring sites of $|\text{AKLT}\rangle$.

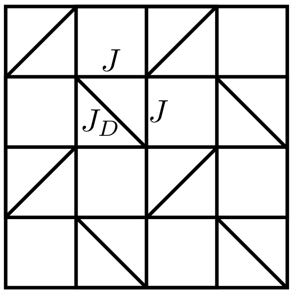
b) **1 Point**

Consider the Majumdar–Ghosh states $|\psi\rangle$ (from Problem **1**) and the state $|\text{AKLT}\rangle$. What could be the elementary excitations (approximately and on the level of these drawings)? Use the constructed "parent" Hamiltonians to retrieve a major difference between these excitations.

c) **2 Points**

Generalize the AKLT idea to two spatial dimensions and draw a non-magnetic, translationally invariant state on the honeycomb lattice. Find the corresponding parent Hamiltonian.

3. Non-Magnetic States III: Shastry–Sutherland **6 Points**



The graph shown in the figure is a variant of the square lattice and called "Shastry–Sutherland lattice". We consider the $S = 1/2$ Heisenberg Hamiltonian

$$\mathcal{H} = J \sum_{\langle ij\rangle_\square} \vec{S}_i \cdot \vec{S}_j + J_D \sum_{\langle ij\rangle_D} \vec{S}_i \cdot \vec{S}_j\,, \tag{4}$$

where the first term describes spin exchange on horizontal and vertical bonds, while the second term on the diagonal bonds.

Find a spin-singlet state consisting of nearest-neighbor singlet bonds which is the groundstate of $\mathcal{H}$. Which ratio J/J_D must be chosen in order to allow for an exact proof? What is the corresponding groundstate energy?

Hint: Consider first an isolated triangle of the Shastry–Sutherland lattice, described by the Hamiltonian $\mathcal{H}^\triangle = I_{12}\,\vec{S}_1 \cdot \vec{S}_2 + I_{23}\,\vec{S}_2 \cdot \vec{S}_3 + I_{31}\,\vec{S}_3 \cdot \vec{S}_1$. Convince yourself that the groundstate of $\mathcal{H}^\triangle$ is given by a singlet bond on two sites of the triangle, and the third site remains "free". Now cover the full lattice with such triangles under the constraint that there must be one singlet bond on each triangle. What is the relation between the I_{ij} and J, J_D in this case?