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Strongly Correlated Electrons
1. Übung

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1. Integral representation of the θ-function **1 point**

Show the validity of the following integral representation of the θ-function:

$$\lim_{\eta \to 0+} \frac{i}{2\pi} \int_{-\infty}^{+\infty} dx \frac{e^{-ixt}}{x + i\eta} = \theta(t). \tag{1}$$

2. Sum rule **2 points**

Prove the following sum rule

$$\int_{-\infty}^{\infty} d\omega \, \left[\omega G^r_{AB}(\omega) - \langle [\hat{A}, \hat{B}]_- \rangle \right] = \pi \langle [\partial_t \hat{A}(t), \hat{B}(0)]_- \rangle |_{t=0}, \tag{2}$$

where $G^r_{AB}(\omega)$ is the Fourier transform of the retarded Green's function

$$G^r_{AB}(t) = -i\theta(t) \langle [\hat{A}(t), \hat{B}(0)]_- \rangle.$$

Use Eq. (2) to discuss the high-frequency behavior of $G^r_{AB}(\omega)$.
(Hint: Derive an equation of motion for $G^r_{AB}(t)$ by considering its time derivative.)

3. Interaction representation **3 points**

The aim of this exercise is to verify the following identity

$$e^{-\beta \hat{H}} = e^{-\beta \hat{H}_0} \hat{T} \exp \left(- \int_0^\beta d\lambda \, \hat{H}_I(-i\lambda) \right), \tag{3}$$

where $\hat{H} = \hat{H}_0 + \hat{H}_I$ is the total Hamiltonian of the system, $\hat{H}_0$ is the unperturbed (solvable) part, and $\hat{H}_I$ is the interaction term. Finally, $\hat{T}$ is the time-ordering operator (see definition below).

a) **1 point**
Start by considering the following operator

$$\hat{\hat{U}}(t) = e^{i\hat{H}_0 t} e^{-i\hat{H}t}, \tag{4}$$

with $\hat{\hat{U}}(t=0) = 1$. Derive the equation of motion (differential equation) for $\hat{\hat{U}}(t)$ and show that it can be written as

$$\hat{\hat{U}}(t) = 1 - i \int_0^t dt_1 \, \hat{H}_I(t_1) \hat{\hat{U}}(t_1), \tag{5}$$

where $\hat{H}_I(t) = e^{i\hat{H}_0 t} \hat{H}_I e^{-i\hat{H}_0 t}$ is the time evolution of $\hat{H}_I(t)$ in the interaction representation.
(Note that $\hat{S}(t, t')$ given in the lecture is related to $\hat{\hat{U}}(t)$ by $\hat{S}(t, t') = \hat{\hat{U}}(t) \hat{\hat{U}}^\dagger(t')$.)

b) **2 points**
Consider now the time-ordering operator $\hat{T}$, which chronologically orders a set of time-dependent operators $\hat{A}(t_i)$, for instance,

$$\hat{T} \left[\hat{A}(t_1) \hat{A}(t_3) \hat{A}(t_4) \hat{A}(t_2) \right] = \hat{A}(t_4) \hat{A}(t_3) \hat{A}(t_2) \hat{A}(t_1) \quad \text{if} \quad t_4 > t_3 > t_2 > t_1. \tag{6}$$

Show that Eq. (5) can be used to write $\hat{\hat{U}}(t)$ as

$$\hat{\hat{U}}(t) = \hat{T} \exp\left(-i \int_0^t dt'\, \hat{H}_I(t')\right). \tag{7}$$

Prove now Eq. (3).

4. Single-site Hubbard model **5 points**

Consider the single-site "Hubbard model"

$$\hat{K} = \hat{H} - \mu\hat{N} = U\hat{n}_\uparrow\hat{n}_\downarrow - \mu \sum_{\sigma=\uparrow,\downarrow} \hat{n}_\sigma, \tag{8}$$

where $\hat{n}_\sigma = \hat{c}_\sigma^\dagger \hat{c}_\sigma$, with $\hat{c}_\sigma^\dagger$ ($\hat{c}_\sigma$) creating (annihilating) an electron with spin $\sigma = \uparrow, \downarrow$ on a single site, is the electron density operator, and $U > 0$ is the on-site repulsion energy. Notice that U is the amount of energy that should be paid if two electrons with different spin are on the same site.

a) **1 point**
Find the eigenstates and the corresponding eigenenergies of the model (8).

b) **2 points**
Calculate the Fourier transform in time, $G_\sigma^r(\omega)$, of the retarded single-particle Green's function $G_\sigma^r(t) = -i\langle[\hat{c}_\sigma(t), \hat{c}_\sigma^\dagger(0)]_+\rangle\,\theta(t)$ at $T = 0$ using the Lehmann representation for the three cases i) $\mu < 0$, ii) $0 < \mu < U$, and iii) $U - \mu < 0 < \mu$. Compare to the free-fermion result.

c) **1 point**
How does the spectral function $A(\omega) = -\frac{1}{\pi}\mathrm{Im}G^r(\omega)$ read ? Discuss the outcome of possible photoemission and inverse photoemission experiments.

d) **1 point**
Calculate the interaction-induced self-energy for the non-trivial case ii).