Technische Universität Dresden
Institut für Theoretische Physik

Prof. Dr. M. Vojta
Dr. T. Meng

Strongly Correlated Electrons
6. Übung

Sommersemester 2015

1. Correlation functions of 1D fermions **5 points**

As discussed in the lecture, the dispersion of one-dimensional (1D) fermions can be linearized close to the Fermi points at momentum $|k| \approx k_F$, where the electrons have a velocity v_F. The fermionic field $\Psi(x)$ can be decomposed as

$$\Psi(x) \approx e^{ik_F x}\Psi_+(x) + e^{-ik_F x}\Psi_-(x), \tag{1}$$

where we have introduced two independent species of fermions, the left $(-)$ and right $(+)$ movers. In this exercise, you will calculate the imaginary time density-density correlation function

$$C(x,\tau) = \langle T_\tau (\Psi_+^\dagger \Psi_-)(x,\tau)(\Psi_-^\dagger \Psi_+)(0,0)\rangle, \tag{2}$$

independently using both the fermionic and the bosonic language. This procedure allows to determine the parameters of the bosonic theory. The first unknown parameter is the prefactor Γ in the bosonization formula

$$\Psi_s^\dagger(x) = \Gamma e^{is\phi(x)} e^{-i\theta(x)}, \tag{3}$$

where $s = \pm$ denotes the direction of motion, while $\phi(x)$ is the bosonic field associated with density fluctuations of the electron gas, $:\rho(x) := -\frac{1}{\pi}\partial_x\phi$ (where $: \ldots :$ denotes normal ordering with respect to the non-interacting vacuum), and $\theta(x)$ is its dual field. These obey $[\phi(x), \theta(x')] = i\pi \mathrm{sgn}(x'-x)/2$. The second free parameter is the coupling constant c which appears in the bosonic action below.

a) **1 point**

Given that the fields $\Psi_s(x)$ are described by the Hamiltonian

$$\mathcal{H} = -iv_F \int dx \left(\Psi_+^\dagger(x)\partial_x\Psi_+(x) - \Psi_-^\dagger(x)\partial_x\Psi_-(x)\right), \tag{4}$$

calculate the correlation function in Eq. (2).
(Hint : the left and right contributions factorize.)

b) **1 point**

In the absence of interactions, an equivalent Hamiltonian to the one given in Eq. (4) is given by

$$\mathcal{H}' = \frac{1}{2c}\int dx \left[(\partial_x\phi(x))^2 + (\partial_x\theta(x))^2\right]. \tag{5}$$

Calculate the correlation functions $\langle T_\tau(\phi(x,\tau) - \phi(0,0))^2\rangle$, $\langle T_\tau(\theta(x,\tau) - \theta(0,0))^2\rangle$ and $\langle T_\tau\theta(x,\tau)\phi(0,0)\rangle$.
(Hint : you can for instance use the equation of motion approach to calculate the Green's functions of ϕ and θ. These obey a 2D Poisson equation. During the evaluation of momentum sums, you may find the introduction of a high-momentum cutoff α^{-1} helpful, where α is some small length that is taken to zero at the end of the calculation.)

c) **1 point**

Using the results you have just obtained, calculate $C(x,\tau)$ in the bosonic language.
(Hint : you may use the identity $\langle 0|e^A|0\rangle = e^{\langle 0|\frac{A^2}{2}|0\rangle}$.)

d) **1 point**

Assuming that both c and Γ are positive real numbers, deduce their values from the comparison of the bosonic and the fermionic calculation. Check that the value of Γ is consistent with the anticommutation relation $\{\Psi_s^\dagger(x), \Psi_s(x')\} = \delta(x-x')$.

e) **1 point**

In the presence of interactions between fermions, the bosonic Hamiltonian is generalized to

$$\mathcal{H} = \frac{1}{2c} \int dx \left[\frac{1}{K} (\partial_x \phi(x))^2 + K (\partial_x \theta(x))^2 \right], \tag{6}$$

where K is a real, and positive number. Recalculate the correlation function $C(x, \tau)$ in the interacting case using a rescaling of the fields θ and ϕ that preserves the commutation relation. (NB: the limit $\alpha \to 0$ is now singular, which is an artifact of not having properly taking into account the finite range of interactions.)