Topological condensed matter physics
Problem set 2
Summer term 2016

1. Domain wall bound state in the SSH-model

   a) 1 Point
   Starting from the time-independent Schrödinger equation of a general second-quantized fermionic $(2 \times 2)$-Hamiltonian,
   \[
   H = \int dx (c_1^\dagger(x), c_2^\dagger(x)) \begin{pmatrix} h_{11}(x) & h_{12}(x) \\ h_{21}(x) & h_{22}(x) \end{pmatrix} \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}
   \]
   where $c_{1,2}(x)$ are annihilation operators, use the ansatz
   \[
   \ket{\Psi} = \int dx \left( u(x) c_1^\dagger(x) + v(x) c_2^\dagger(x) \right) \ket{0},
   \]
   where $\ket{0}$ is the vacuum defined by $c_{1,2}(x)\ket{0} = 0$, to obtain a matrix equation for the coefficients $u(x)$ and $v(x)$.

   b) 1 Point
   For one spin species, a continuum version of the (infinitely long) SSH model is described by the Hamiltonian
   \[
   H_{SSH} = \int dx \Psi^\dagger(x)(-iv_F \partial_x \sigma_x + m(x) \sigma_y)\Psi(x)
   \]
   (in the lecture, the Fermi velocity $v_F$ and mass $m$ were given by $v_F = -ta$ and $m = 2\delta t$). $\Psi(x)$ is a spinor of two different annihilation operators. Assuming that the mass is a monotonically increasing function with a sign change at $x = 0$,
   \[
   m(x < 0) < 0 , \quad m(x = 0) = 0 , \quad m(x > 0) > 0
   \]
   find the zero-energy bound state(s) associated with the domain wall. How many are there?

2. Time-reversal (TR) symmetry

   a) 1 Point
   Consider a single band for spinless fermions below the Fermi energy with Bloch state $\ket{u(k)}$. Show that Hall conductance is zero for the case of TR invariance.

   b) 1 Point
   Consider the Hamiltonian for spinful fermions $H = \sum_{k,\sigma,\sigma'} c_{k\sigma}^\dagger h_{\sigma\sigma'}(k)c_{k\sigma'}$. How does the Bloch matrix $h_{\sigma\sigma'}(k)$ transform if $H$ is TR-invariant, i.e., $H = THT^{-1}$?
3. π-flux bands 6 Points

Consider a tight-binding model for spinless fermions on a 2D square lattice subject to a uniform magnetic field. The corresponding vector potential is given by

\[ A(x) = \frac{hc}{e} \frac{\pi}{a^2} y \hat{x} \]

The Hamiltonian for such a model governed by

\[ H = \sum_{\langle lm \rangle} \left( t_{lm} c_l^\dagger c_m + \text{h.c.} \right) \]

Each pair of nearest neighbor sites \( \langle lm \rangle \) in the sum appears only once. The hopping amplitude reads

\[ t_{lm} = t_0 \exp \left( \frac{ie}{hc} \int_{m}^{n} ds \, A(x) \right) \]

with \( t_0 \) being constant and \( a \) the lattice spacing.

a) 1 Point
Show that the magnetic flux per plaquette (square of four adjacent sites) is given by \( \Phi_0 / 2 \) where \( \Phi_0 = \frac{hc}{e} \) is the Dirac flux quantum.

b) 2 Points
Sketch the lattice including the phases of the hopping amplitudes \( t_{lm} \) of neighboring lattice sites. What is the minimal number of lattice sites in the elementary unit cell? Hint: consider the periodicity of the hopping phases under translations in the different directions.

Now choose a primitive unit cell and draw the corresponding Brillouin zone. How many energy bands do you expect to obtain? And how many allowed values of Bloch momentum \( \mathbf{q} \) are there for a lattice with \( N \) sites (PBCs imposed)?

c) 3 Points
Perform a Fourier transform and show that the Hamiltonian can now be written as

\[ H = \sum_q \left( c_{q,1}^\dagger, c_{q,2}^\dagger \right) H_q \left( c_{q,1}, c_{q,2} \right) \]

Determine the matrix elements of \( H_q \) for this π-flux lattice.

Show that the coefficients

\[ u_q = \begin{pmatrix} u_{q,1} \\ u_{q,2} \end{pmatrix} \]

of the eigenstates \( |\psi_q\rangle = \left( c_{q,1}^\dagger, c_{q,2}^\dagger \right) u_q |0\rangle \)

are given by the eigenvalue equation \( (H_q - E_q) u_q = 0 \).

Express \( H_q \) as a superposition of Pauli matrices and show that the previous eigenvalue equation represents a Dirac equation, i.e., \( H_q^2 \) is diagonal. Compute the eigenvalues \( E_{q,n} \) and eigenvectors \( u_{q,n} \).

Sketch the bandstructure and try to identify degeneracy points in the Brillouin zone. Expand the spectrum around these special points and conjecture about the topological properties of the π-flux lattice (Berry phase, Hall conductance, etc.).