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Topological condensed matter physics
Problem set 2

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1. Domain wall bound state in the SSH-model — 2 Points

a) **1 Point**

Starting from the time-independent Schrödinger equation of a general second-quantized fermionic (2×2)-Hamiltonian,

$$H = \int dx (c_1^\dagger(x), c_2^\dagger(x)) \begin{pmatrix} h_{11}(x) & h_{12}(x) \\ h_{21}(x) & h_{22}(x) \end{pmatrix} \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} \tag{1}$$

where $c_{1,2}(x)$ are annihilation operators, use the ansatz

$$|\Psi\rangle = \int dx \, \Big(u(x) c_1^\dagger(x) + v(x) c_2^\dagger(x) \Big) |0\rangle, \tag{2}$$

where $|0\rangle$ is the vacuum defined by $c_{1,2}(x)|0\rangle = 0$, to obtain a matrix equation for the coefficients $u(x)$ and $v(x)$.

b) **1 Point**

For one spin species, a continuum version of the (infinitely long) SSH model is described by the Hamiltonian

$$H_{\mathrm{SSH}} = \int dx \, \Psi^\dagger(x)(-iv_F \partial_x \sigma_x + m(x)\sigma_y)\Psi(x) \tag{3}$$

(in the lecture, the Fermi velocity v_F and mass m were given by $v_F = -ta$ and $m = 2\delta t$). $\Psi(x)$ is a spinor of two different annihilation operators. Assuming that the mass is a monotonically increasing function with a sign change at $x = 0$,

$$m(x < 0) < 0 \quad , \quad m(x = 0) = 0 \quad , \quad m(x > 0) > 0 \; , \tag{4}$$

find the zero-energy bound state(s) associated with the domain wall. How many are there?

2. Time-reversal (TR) symmetry — 2 Points

a) **1 Point**

Consider a single band for spinless fermions below the Fermi energy with Bloch state $|u(\boldsymbol{k})\rangle$. Show that Hall conductance is zero for the case of TR invariance.

b) **1 Point**

Consider the Hamiltonian for spinful fermions $H = \sum_{\boldsymbol{k},\sigma\sigma'} c^\dagger_{\boldsymbol{k}\sigma} h_{\sigma\sigma'}(\boldsymbol{k}) c_{\boldsymbol{k}\sigma'}$. How does the Bloch matrix $h_{\sigma\sigma'}(\boldsymbol{k})$ transform if H is TR-invariant, *i.e.*, $H = THT^{-1}$?

3. π-flux bands **6 Points**

Consider a tight-binding model for spinless fermions on a 2D square lattice subject to a uniform magnetic field. The corresponding vector potential is given by

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\hbar c}{e}\frac{\pi}{a^2} y \hat{\boldsymbol{x}} \ .$$

The Hamiltonian for such a model governed by

$$H = \sum_{\langle lm \rangle} \left(t_{lm}\, c_l^\dagger c_m + \text{h.c.} \right) .$$

Each pair of nearest neighbor sites $\langle lm \rangle$ in the sum appears only once. The hopping amplitude reads

$$t_{lm} = t_0 \exp\left(\frac{ie}{\hbar c} \int_l^m d\boldsymbol{s}\, \boldsymbol{A}(\boldsymbol{x}) \right)$$

with t_0 being constant and a the lattice spacing.

a) **1 Point**

Show that the magnetic flux per plaquette (square of four adjacent sites) is given by $\Phi_0/2$ where $\Phi_0 = \frac{hc}{e}$ is the Dirac flux quantum.

b) **2 Points**

Sketch the lattice including the phases of the hopping amplitudes t_{lm} of neighboring lattice sites. What is the minimal number of lattice sites in the elementary unit cell? *Hint:* consider the periodicity of the hopping phases under translations in the different directions.

Now choose a primitive unit cell and draw the corresponding Brillouin zone. How many energy bands do you expect to obtain? And how many allowed values of Bloch momentum $\boldsymbol{q}$ are there for a lattice with N sites (PBCs imposed)?

c) **3 Points**

Perform a Fourier transform and show that the Hamiltonian can now be written as

$$H = \sum_{\boldsymbol{q}} \left(c_{\boldsymbol{q},1}^\dagger, c_{\boldsymbol{q},2}^\dagger \right) H_{\boldsymbol{q}} \begin{pmatrix} c_{\boldsymbol{q},1} \\ c_{\boldsymbol{q},2} \end{pmatrix} .$$

Determine the matrix elements of $H_{\boldsymbol{q}}$ for this π-flux lattice.

Show that the coefficients

$$\boldsymbol{u}_{\boldsymbol{q}} = \begin{pmatrix} u_{\boldsymbol{q},1} \\ u_{\boldsymbol{q},2} \end{pmatrix} \quad \text{of the eigenstates} \quad |\psi_{\boldsymbol{q}}\rangle = \left(c_{\boldsymbol{q},1}^\dagger, c_{\boldsymbol{q},2}^\dagger \right) \boldsymbol{u}_{\boldsymbol{q}} \,|\, 0 \,\rangle$$

are given by the eigenvalue equation $(H_{\boldsymbol{q}} - E_{\boldsymbol{q}})\, \boldsymbol{u}_{\boldsymbol{q}} = 0$.

Express $H_{\boldsymbol{q}}$ as a superposition of Pauli matrices and show that the previous eigenvalue equation represents a Dirac equation, *i.e.*, $H_{\boldsymbol{q}}^2$ is diagonal. Compute the eigenvalues $E_{\boldsymbol{q},n}$ and eigenvectors $\boldsymbol{u}_{\boldsymbol{q},n}$.

Sketch the bandstructure and try to identify degeneracy points in the Brillouin zone. Expand the spectrum around these special points and conjecture about the topological properties of the π-flux lattice (Berry phase, Hall conductance, etc.).