1. Four ways to solve the Kitaev chain 4 Points

Consider a simple version of the Kitaev chain with only two sites, and described by the Hamiltonian

\[ H = \epsilon_1 c_1^\dagger c_1 + \epsilon_2 c_2^\dagger c_2 - t(c_1^\dagger c_2 + c_2^\dagger c_1) + \Delta(c_1^\dagger c_2^\dagger + c_2 c_1) \]  

where \( c_i \) annihilates an electron on site \( i = 1, 2 \), the on-site energy is \( \epsilon_i \), \( t > 0 \) is the hopping between the two sites, and \( \Delta > 0 \) is the superconducting order parameter.

a) 1 Point

Find the eigenenergies of the Hamiltonian by considering matrix-elements of \( H \) using the basis states

\[ |00\rangle = |0\rangle, \quad |10\rangle = c_1^\dagger |0\rangle, \quad |01\rangle = c_2^\dagger |0\rangle, \quad |11\rangle = c_1^\dagger c_2^\dagger |0\rangle, \]  

where the vacuum \( |0\rangle \) satisfies \( c_{1,2} |0\rangle = 0 \). What distinguishes the two sectors that the Hamiltonian can be decomposed into? Which states span the ground state manifold at the sweet spot \( \epsilon_1 = \epsilon_2 = 0, t = \Delta \), and which states are then excited states?

b) 1 Point

Next, find the Bogoliubov-de Gennes Hamiltonian associated with \( H \), and relate its eigenvalues to the energies found in a).

c) 1 Point

For \( t = 0 \), you can alternatively rewrite the Hamiltonian with a Nambu spinor as

\[ H = (c_1^\dagger, c_2^\dagger) \mathcal{H} (c_1, c_2^\dagger) + E_0. \]  

What is the vacuum associated with this form of the Hamiltonian? Find \( \mathcal{H} \) and \( E_0 \), calculate the spectrum of \( H \), and relate the results to part a).

d) 1 Point

Finally, introduce Majorana fermions \( \gamma_j^{(1,2)} \) as

\[ c_j = (\gamma_j^{(1)} + i\gamma_j^{(2)})/2. \]  

Rewrite the Hamiltonian in the Majorana language. What happens at the sweet spot \( \epsilon_1 = \epsilon_2 = 0 \) and \( t = \Delta (> 0) \)? Define a complex fermionic zero-mode, and re-express the associated annihilation and creation operator in terms of the original \( c_j \)-operators. Compare with the results of part a).
2. Braiding of Majorana zero modes 4 Points

Besides being topological zero energy bound states, the Majorana zero modes in the Kitaev chain are interesting because they show non-Abelian braiding. In short, this means that when a Majorana is braided around another, the state of the system can change. To illustrate this concept, consider a system that combines four such Majorana zero modes on some sort of “tracks” allowing to exchange two neighboring Majoranas in the following way:

a) While particle number is not conserved in a Majorana system, the fermionic parity is a good quantum number (even or odd number of fermions in the system correspond to parity eigenvalues +1 and -1, respectively). To define the parity, however, one needs proper fermionic states. Using $c = (\gamma_1 + i\gamma_2)/2$ and $d = (\gamma_3 + i\gamma_4)/2$, what is the total parity of system?

b) Without being specific about the details of the process, the exchange of Majoranas 2 and 3 can be understood as an adiabatic evolution of the system upon changing some microscopic parameters, and is hence represented by a unitary matrix $U_{23}$. Considering furthermore that different Majoranas do not interact, the evolution operator $U_{23}$ can only involve the Majorana fermion operators $\gamma_2$ and $\gamma_3$ since only these Majoranas are affected by the braiding. Based on these remarks, show that the most general form of $U_{23}$ is

$$U_{23} = e^{i\varphi} (\sin(x) + \cos(x)\gamma_2\gamma_3) \quad (5)$$

with real numbers $\varphi$ and $x$ (Hint: you may assume that the adiabatic evolution does not affect the parity of the system).

c) Why is a reasonable choice for $x$ is given by $x = \pm\pi/4$? How can you interpret the two signs of $x$?

d) Now consider a full circular braiding of $\gamma_2$ once around $\gamma_3$. What is the operator corresponding to this braiding? How does it affect the parity of the $c$-mode, the parity of the $d$-mode, and the total parity? What does this mean if the two parities are used as logical states for qubits?