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Topological condensed matter physics
Problem set 5

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1. Four ways to solve the Kitaev chain **4 Points**

Consider a simple version of the Kitaev chain with only two sites, and described by the Hamiltonian

$$H = \epsilon_1 c_1^\dagger c_1 + \epsilon_2 c_2^\dagger c_2 - t(c_1^\dagger c_2 + c_2^\dagger c_1) + \Delta(c_1^\dagger c_2^\dagger + c_2 c_1) \quad (1)$$

where c_i annihilates an electron on site $i = 1, 2$, the on-site energy is ϵ_i, $t > 0$ is the hopping between the two sites, and $\Delta > 0$ is the superconducting order parameter.

a) **1 Point**

Find the eigenenergies of the Hamiltonian by considering matrix-elements of H using the basis states

$$|00\rangle = |0\rangle \quad , \quad |10\rangle = c_1^\dagger|0\rangle \quad , \quad |01\rangle = c_2^\dagger|0\rangle \quad , \quad |11\rangle = c_1^\dagger c_2^\dagger|0\rangle \ , \quad (2)$$

where the vacuum $|0\rangle$ satisfies $c_{1,2}|0\rangle = 0$. What distinguishes the two sectors that the Hamiltonian can be decomposed into? Which states span the ground state manifold at the sweet spot $\epsilon_1 = \epsilon_2 = 0$, $t = \Delta$, and which states are then excited states?

b) **1 Point**

Next, find the Bogoliubov-de Gennes Hamiltonian associated with H, and relate its eigenvalues to the energies found in a).

c) **1 Point**

For $t = 0$, you can alternatively rewrite the Hamiltonian with a Nambu spinor as

$$H = (c_1^\dagger, c_2)\mathcal{H}\begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} + E_0. \quad (3)$$

What is the vacuum associated with this form of the Hamiltonian? Find $\mathcal{H}$ and E_0, calculate the spectrum of H, and relate the results to part a).

d) **1 Point**

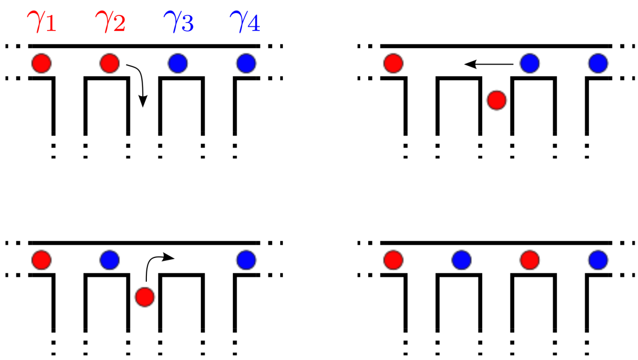
Finally, introduce Majorana fermions $\gamma_j^{(1,2)}$ as

$$c_j = (\gamma_j^{(1)} + i\gamma_j^{(2)})/2. \quad (4)$$

Rewrite the Hamiltonian in the Majorana language. What happens at the sweet spot $\epsilon_1 = \epsilon_2 = 0$ and $t = \Delta\,(> 0)$? Define a complex fermionic zero-mode, and re-express the associated annihilation and creation operator in terms of the original c_j-operators. Compare with the results of part a).

2. Braiding of Majorana zero modes **4 Points**

Besides being topological zero energy bound states, the Majorana zero modes in the Kitaev chain are interesting because they show non-Abelian braiding. In short, this means that when a Majorana is braided around another, the state of the system can change. To illustrate this concept, consider a system that combines four such Majorana zero modes on some sort of "tracks" allowing to exchange two neighboring Majoranas in the following way:



a) **1 Point**

While particle number is not conserved in a Majorana system, the fermionic parity is a good quantum number (even or odd number of fermions in the system correspond to parity eigenvalues +1 and -1, respectively). To define the parity, however, one needs proper fermionic states. Using $c = (\gamma_1 + i\gamma_2)/2$ and $d = (\gamma_3 + i\gamma_4)/2$, what is the total parity of system?

b) **1 Point**

Without being specific about the details of the process, the exchange of Majoranas 2 and 3 can be understood as an adiabatic evolution of the system upon changing some microscopic parameters, and is hence represented by a unitary matrix U_{23}. Considering furthermore that different Majoranas do not interact, the evolution operator U_{23} can only involve the Majorana fermion operators γ_2 and γ_3 since only these Majoranas are affected by the braiding. Based on these remarks, show that the most general form of U_{23} is

$$U_{23} = e^{i\varphi} \left(\sin(x) + \cos(x)\gamma_2\gamma_3 \right) \tag{5}$$

with real numbers φ and x (Hint: you may assume that the adiabatic evolution does not affect the parity of the system).

c) **1 Point**

Why is a reasonable choice for x is given by $x = \pm\pi/4$? How can you interpret the two signs of x?

d) **1 Point**

Now consider a full circular braiding of γ_2 once around γ_3. What is the operator corresponding to this braiding? How does it affect the parity of the c-mode, the parity of the d-mode, and the total parity? What does this mean if the two parities are used as logical states for qubits?