Exercises for "Quantum Phase Transitions" SS 16

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Exercise 1
(for 07.04.16, 11:10)

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1. Landau functional for a first-order phase transition

Consider the free-energy density

$$f(\varphi) = f_\mathrm{n} + f_0 \left[\frac{a}{2}\varphi^2 + \frac{b}{4}\varphi^4 + \frac{c}{6}\varphi^6\right], \tag{1}$$

which depends on the real order parameter φ, a depends on the temperature, b and c are temperature-independent, and $b < 0$, $c > 0$.

(a) Determine the extrema of the functional (1). List all possibilities and sketch $f(\varphi) - f_\mathrm{n}$ in each case.

(b) Calculate the critical value a_c of the parameter a where the position $\varphi_\mathrm{min}(a)$ of the global minimum of (1) changes discontinuously.

(c) Sketch the free energy $f(\varphi_\mathrm{min})$ as a function of the parameter a in the vicinity of the phase transition. Why is it a first-order phase transition?
Hint: Expand $f(\varphi_\mathrm{min})$ up to first order in $\delta a = a - a_\mathrm{c}$ around $\delta a = 0$.

2. Two order parameters

Determine the phase diagram of a system with two real order parameters φ_1 und φ_2, whose free-energy density is given by

$$f(\varphi_1, \varphi_2) = \frac{r}{2}\left(\varphi_1^2 + \varphi_2^2\right) - \frac{g}{2}\left(\varphi_1^2 - \varphi_2^2\right) + \frac{u}{4}\left(\varphi_1^4 + \varphi_2^4\right) + \frac{v}{2}\varphi_1^2\varphi_2^2, \tag{2}$$

where $u, v > 0$.

(a) Start by determining all extrema of the functional (2). Which values are taken by φ_1^2, φ_2^2 at these extrema?

(b) Which conditions have to be posed on φ_1^2 and φ_2^2? Discuss which phases (i.e., configurations of φ_1 and φ_2) are physically reasonable in which areas of the (r, g) plane.

(c) In each case, determine the phase with the lowest free energy as function of r and g. Distinguish between $u^2 < v^2$ and $u^2 > v^2$.

(d) What is the order of the phase transitions?

(e) Sketch the phase diagram in the (r, g) plane for both $u^2 < v^2$ and $u^2 > v^2$.