Exercises for "Quantum Phase Transitions" SS 16

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Exercise 2
(for 21.04.16, 11:10)

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1. Tricritical Point in an Antiferromagnet (6 points)

An external magnetic field h in an antiferromagnet couples to the magnetization m rather than to the antiferromagnetic order parameter, the staggered magnetization n. Assume that the coupling between n and m is described phenomenologically via the free energy

$$f(n,m) = \frac{t}{2}n^2 + \frac{b}{4}n^4 + \frac{v}{2}m^2 + \frac{w}{2}n^2m^2 - hm, \tag{1}$$

where $t = a\cdot(T - T_0)$, and a, b, v, w are positive and independent of temperature.

(a) Show that this model has a paramagnetic phase with magnetization m_0. Determine the relation between the magnetization and the staggered magnetization, *i.e.,* $m(n^2)$, in the antiferromagnetic phase.

(b) Consider the antiferromagnetic phase near the phase transition, *i.e.,* for small values of n^2. Write $m = m_0 + \delta m$, expand $m(n^2)$ for small n^2, and derive a relation between δm and n^2.

(c) Evaluate the effective free energy density for the staggered magnetization

$$g(n) = f(n, m_0 + \delta m) - f(0, m_0). \tag{2}$$

(d) Show that the model (1) has a tricritical point at temperature T_t and field h_t, where

$$T_\mathrm{t} = T_0 - \frac{bv}{2aw}, \quad h_\mathrm{t}^2 = \frac{bv^3}{2w^2}. \tag{3}$$

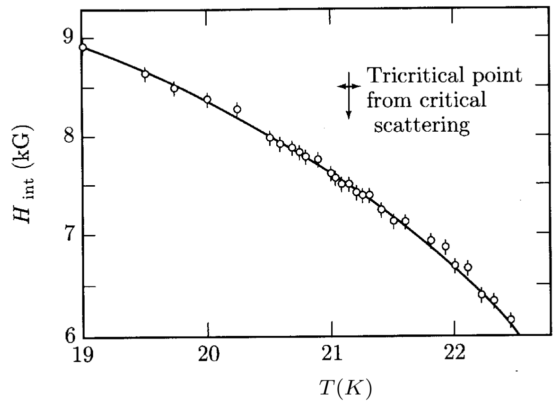
Hint: At a tricritical point first and second order transition lines meet. Depending on the sign of the coefficient of n^4, $g(n)$ describes a first (second) order phase transition between the paramagnetic and antiferromagnetic phase.

(e) Show that the mean-field phase diagram is similar to the one of the metamagnet $FeCl_2$ shown below, where the second-order transition temperature for $h < h_\mathrm{t}$ is

$$T_\mathrm{c} = T_0 - \frac{wh^2}{av^2}, \tag{4}$$

and the first-order transition temperature for $h > h_\mathrm{t}$ is

$$T_\mathrm{c} = T_0 - \frac{3wh^2}{4av^2} - \frac{bv}{4aw} + \frac{b^2v^4}{16aw^3h^2}. \tag{5}$$



Experimental phase diagram for $FeCl_2$;
the tricritical point is located at $T_t \approx 21.2$ K, $h_t \approx 7.4$ kG.

2. Static Scaling Hypothesis (4 points)

In this problem, we assume the static scaling hypothesis discussed in class.

(a) Derive the relation

$$\delta = \frac{d+2-\eta}{d-2+\eta} \tag{6}$$

between the critical isotherm exponent δ and the correlation function exponent η.

Hint: Use the set of expressions derived from the correlation function scaling which relate y_t and y_h to the other critical exponents.

(b) In principle, critical exponents could be different above and below a transition. We wish to show that $\nu(T > T_c)$ and $\nu'(T < T_c)$ are equal.

Hint: The scaling hypothesis leads to the relation for the singular part of the free energy density

$$f_s(t,h) = |t|^{d\bar{\nu}}\, F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu} y_h}}\right), \tag{7}$$

with $\bar{\nu} = \nu$ for $t > 0$ and $\bar{\nu} = \nu'$ for $t < 0$. For fixed $h \neq 0$, $f_s(t,h)$ should be a smooth function of t, because the only singularity which we expect is at $t = h = 0$. Show that $f_s(t,h)$ can be written in the form

$$f_s(t,h) = h^{d/y_h}\, \phi^{\pm}\left(\frac{h}{|t|^{\bar{\nu} y_h}}\right), \tag{8}$$

and explain how the smoothness assumption mentioned above constrains the analytic form of the functions $\phi^{\pm}$.