Exercises for "Quantum Phase Transitions" SS 16

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Exercise 3
(for 09.05.16, 11:10)

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1. Partition function for noninteracting fields (4 points)

Show that the M-dimensional Gaussian integral (discretized partition function) is

(a) for a real boson field φ and symmetric and real matrix $S = S^{\mathrm{T}}$:

$$Z_0 = \prod_{k=1}^{M} \int \frac{d\varphi_k}{\sqrt{2\pi}} \exp\left(-\sum_{j,k=1}^{M} \frac{1}{2}\varphi_j S_{jk}\varphi_k\right) = (\det S)^{-1/2}, \tag{1}$$

(b) for complex boson fields ϕ^*, ϕ and Hermitian matrix $S = S^\dagger$:

$$Z_0 = \prod_{k=1}^{M} \int \frac{d\phi_k^* d\phi_k}{2\pi i} \exp\left(-\sum_{j,k=1}^{M} \phi_j^* S_{jk}\phi_k\right) = (\det S)^{-1}, \tag{2}$$

(c) for Grassmann fields $\psi^\dagger$, ψ and Hermitian matrix $S = S^\dagger$:

$$Z_0 = \prod_{k=1}^{M} \int d\psi_k^\dagger d\psi_k \exp\left(-\sum_{j,k=1}^{M} \psi_j^\dagger S_{jk}\psi_k\right) = \det S. \tag{3}$$

Hint: Recall the integration rules for Grassmann variables ξ: $\int d\xi\, \xi = 1$ and $\int d\xi\, 1 = 0$.

2. Partition function and the continuum limit: right and wrong! (6 points)

In this exercise you are supposed to get familiar with some subtleties of the continuum limit for coherent-state path integrals that are sometimes hidden under the carpet ...

(a) The standard representation of the partition function for noninteracting bosons (fermions) is

$$Z_0 = \lim_{M\to\infty} \prod_\alpha \left[\prod_{k=1}^{M} \int \frac{d\phi_k^* d\phi_k}{\mathcal{N}} \exp\left(-\sum_{j,k=1}^{M} \phi_j^* S_{jk}^\alpha \phi_k\right)\right]. \tag{4}$$

Here α labels generically the single-particle eigenstates of the system. The matrix S^α is given by

$$S^\alpha = \begin{pmatrix} 1 & 0 & \dots & 0 & -\zeta a \\ -a & 1 & 0 & \ddots & 0 \\ 0 & -a & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & -a & 1 \end{pmatrix} \tag{5}$$

where we use the shorthand $a = 1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)$ and $\zeta = 1$ ($\zeta = -1$) for bosons (fermions) with normalization $\mathcal{N} = 2\pi i$ ($\mathcal{N} = 1$). Derive the well-known formula for the partition function

$$Z_0 = \prod_\alpha \big(1 - \zeta e^{-\beta(\epsilon_\alpha - \mu)}\big)^{-\zeta}. \tag{6}$$

In addition, use the path integral to calculate the two-point (imaginary) time-ordered Green's function $G^0(\alpha\tau_1|\alpha'\tau_2) = -\langle \mathrm{T} a_\alpha(\tau_1) a^\dagger_{\alpha'}(\tau_2)\rangle$.

(b) The continuum limit of S^α, given by $S^\alpha_{\text{cont.}} = \frac{\partial}{\partial\tau} - \mu + \epsilon_\alpha$, admits different discrete representations. Suppose now we start with another discrete matrix $\tilde{S}^\alpha$ (whose continuum limit is also $S^\alpha_{\text{cont.}}$) given by

$$\tilde{S}^\alpha = \begin{pmatrix} -a & 1 & \ldots & 0 & 0 \\ 0 & -a & 1 & \ddots & 0 \\ \vdots & 0 & -a & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \zeta & \cdots & \cdots & 0 & -a \end{pmatrix}. \tag{7}$$

Already from the trace you should infer that something is wrong in this expression. Does $\tilde{S}$ produce the correct partition function? What about its eigenvalues?