Exercises for "Quantum Phase Transitions"   SS 16

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Exercise 4
(for 02.06.16, 11:10)

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1. Scaling dimensions of higher-order operators (3 points)

Consider the following higher-order perturbations to the Gaussian action

$$S_0 = \int d^d x \, \frac{1}{2} \left[(\nabla\phi)^2 + r\phi^2 \right]$$

of the $O(1)$ model (i.e., ϕ represents a single real scalar boson):

(a) $S = S_0 + u_6 \int d^d x \, \phi^6$

(b) $S = S_0 + u_{4,2} \int d^d x \, (\nabla\phi)^2 \phi^2$

(c) $S = S_0 + u_{k+l,n} \int d^d x \, \underbrace{(\nabla^{n_1}\phi) \cdots (\nabla^{n_k}\phi)}_{k \text{ times}} \phi^l$, with $k, l, \frac{n_i}{2} \in \mathbb{N}$ and $n = \sum_i n_i$

What are the canonical scaling dimensions of the couplings u_6, $u_{4,2}$, and $u_{k+l,n}$? Write down the "tree-level" (i.e., zeroth-order) form of the RG flow of these couplings.

2. Anisotropic perturbation to the $O(2)$ Wilson-Fisher fixed point (7 points)

Consider the $O(2)$ model for the two-component boson field $\phi = (\phi_1, \phi_2)$ (with ϕ_1, ϕ_2 being real scalars) in the presence of an anisotropic perturbation:

$$S = \int d^d x \left[\frac{1}{2}(\nabla\phi_1)^2 + \frac{1}{2}(\nabla\phi_2)^2 + \frac{r}{2}(\phi_1^2 + \phi_2^2) + \frac{u}{4!}(\phi_1^4 + \phi_2^4) + \frac{2v}{4!}\phi_1^2\phi_2^2 \right]. \qquad (1)$$

For $v \neq u$ the continuous $O(2)$ rotational symmetry is explicitly broken, but a residual $\mathbb{Z}_4$ symmetry (fourfold rotations by integer multiples of $\pi/2$) remains intact.

(a) Classify all possible symmetry-allowed operators with respect to their scaling dimension. Are there any relevant or marginal operators near $d = 4$ dimensions that have been omitted in Eq. (1)?

(b) Show that the one-loop RG flow of the suitably rescaled couplings in $d = 4 - \epsilon$ can be written as:

$$\frac{dr}{d\ell} = 2r + \frac{1}{2}u + \frac{1}{6}v \qquad (2)$$

$$\frac{du}{d\ell} = \epsilon u - \frac{3}{2}u^2 - \frac{1}{6}v^2 \qquad (3)$$

$$\frac{dv}{d\ell} = \epsilon v - \frac{2}{3}v^2 - uv \qquad (4)$$

(c) Show that these equations reduce to the expected flow equations of the $O(2)$ model in the limit $u = v$.

(d) Determine the linearized RG flow in the vicinity of the Wilson-Fisher fixed point at $r = r^*$ and $u = v = u^*$:

$$\frac{d\delta g_i}{d\ell} = \sum_{j=1}^{3} B_{ij}\delta g_j + \mathcal{O}(\delta g^2), \qquad \delta g_i \equiv g_i - g_i^*,$$

with the "stability matrix" $B_{ij} = \frac{\partial(dg_i/d\ell)}{\partial g_j}\Big|_{g=g^*}$ and $(g_i) \equiv (r, u, v)$. Is the $\mathbb{Z}_4$ anisotropy $\propto u - v$ relevant or irrelevant at the Wilson-Fisher fixed point? What is the corresponding eigenvalue of the stability matrix?