Exercises for "Quantum Phase Transitions" SS 16

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Exercise 7
(for 14.07.16, 11:10)

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1. Shift exponent in the quantum ϕ^4 theory (4 points)

The ϕ^4 field theory with the action

$$S = \int d^d x \, d\tau \left\{ \frac{1}{2} \left[c^2 (\nabla \phi_\alpha)^2 + (\partial_\tau \phi_\alpha)^2 + r_0 \phi_\alpha^2 \right] + \frac{u_0}{4!} (\phi_\alpha^2)^2 \right\}, \tag{1}$$

($\alpha = 1, 2, \ldots, N$), has a quantum phase transition at $T = 0$, $r_0 = r_\mathrm{c}$. The shift exponent ψ is defined via the temperature-dependent phase boundary

$$T_\mathrm{c} \sim (r_\mathrm{c} - r_0)^\psi, \tag{2}$$

where T_c is the critical temperature. To calculate T_c, note that the phase transition occurs when the renormalized temperature-dependent mass $r(T)$ of the order parameter vanishes. The upper critical dimension for the quantum phase transition is $d_c^+ = 4 - z = 3$.

(a) Below the upper critical dimension d_c^+, use a simple scaling argument to relate ψ to other critical exponents.

(b) For $d > d_\mathrm{c}^+$, the naive scaling analysis above becomes invalid. However, a perturbative calculation of $r(T)$ becomes feasible. To this end, calculate the self-energy of the ϕ propagator in bare perturbation theory to first order in u_0. The temperature dependence of $r(T)$ at $r_0 = r_\mathrm{c}$ allows to obtain ψ in this case.

(c) Apply the procedure of (b) to a situation with $z = 2$ where the bare propagator is $G_\phi^{-1} = i\omega_n - c^2 \vec{k}^2 - r_0$ (instead of $G_\phi^{-1} = -\omega_n^2 - c^2 \vec{k}^2 - r_0$).

Hint to (b) and (c): The shift exponent can be expressed in terms of the dimension d and the dynamical exponent z only.

2. SM-to-CDW transition in graphene (6 points)

Interacting spinless fermions that hop on the honeycomb lattice undergo a quantum phase transition between the weakly-interacting semimetallic (SM) state and an insulating charge-density-wave state (CDW) upon increasing the nearest-neighbor density-density interaction. A Lorentz-invariant effective field theory that describes the universality class of this transition is given by

$$S = \int d^d x \, d\tau \left\{ \bar{\Psi}_i \left(\vec{\sigma} \cdot \nabla + \sigma_3 \partial_\tau \right) \Psi_i + \frac{1}{2} \phi (-\partial_\tau^2 - \nabla^2 + r_0) \phi + g_0 \phi \bar{\Psi}_i \Psi_i + \frac{u_0}{4!} \phi^4 \right\}, \tag{3}$$

in $d = 2$. The Dirac spinor Ψ_i and its Dirac adjoint $\bar{\Psi}_i = \Psi_i^\dagger \sigma_3$ have two components and a "valley" index $i = 1, \ldots, N$. (The physical situation relevant for graphene would be given by $N = 2$, corresponding to the two Dirac points in the Brillouin zone.) The goal of this exercise is to compute the quantum critical behavior in fixed $d = 2$ within the $1/N$ expansion, which is complementary to the ϵ expansion.

(a) Compute scaling dimensions (tree-level terms) and draw Feynman diagrams at one loop to argue that the structure of the RG flow upon integrating out the modes from Λ to Λ/s can be written as

$$\frac{dr}{ds} = (2 - \eta_\phi)r + a_1(N)g^2 + a_2(N)\lambda, \tag{4}$$

$$\frac{d\lambda}{ds} = (4 - d - z - 2\eta_\phi)\lambda + a_3(N)\lambda^2, \tag{5}$$

$$\frac{dg^2}{ds} = (4 - d - z - \eta_\phi - 2\eta_\Psi)g^2 + a_4(N)g^4, \tag{6}$$

$$\eta_\phi = a_5(N)g^2, \tag{7}$$

$$\eta_\Psi = a_6(N)g^2, \tag{8}$$

with combinatorial prefactors $a_1, \ldots, a_6$ which are functions of N only. η_ϕ and η_Ψ are the bosonic and fermionic anomalous dimensions, respectively. Note that η_ϕ and η_Ψ are not necessarily small, in contrast to situation within the ϵ expansion.

(b) How do the coefficients a_i depend on N to leading order in $1/N$?

(c) Find the RG stable fixed point and the critical exponents z, η_ϕ, η_ψ, and ν that describe the universal behavior in the vicinity of the quantum phase transition in $d = 2$ to leading order in $1/N$. Is the one-loop expansion in $d = 2$ reliable at large N?

Hint: Throughout this exercise there is no need to compute the coefficients a_i explicitly!