Technische Universität Dresden — Institut für Theoretische Physik

Exercises for "Theoretical Physics (Master)" **Winter 26/27**

Dr. L. Janssen
Prof. Dr. M. Vojta

Exercise 2 (03.-12.11.26)

---

1. Freedom of choice in infinitesimal time evolution

Consider a system described by a time-independent Hamiltonian $\hat{H}$, whose time evolution is governed by the Schrödinger equation. The aim of this exercise is to derive the corresponding path integral in a general form.

(a) Determine the expression for the operator $\hat{U}(t, t_0)$ that describes the evolution of a state between the times t_0 and t. Then divide the time interval $\Delta t = t - t_0$ into N subintervals δt of equal length, and express the total evolution in terms of the N partial evolutions.
Result: $\hat{U}(t, t_0) = \prod_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\hat{H}\Delta t/N}$

(b) Show that you only need to compute the time evolution in each subinterval correctly up to order $1/N$ if you take the limit $N \to \infty$ at the end of the calculation.
Hint: For large z you can use $z! \approx \sqrt{2\pi z}\,(z/\mathrm{e})^z$. Furthermore, note that $(a+b)^c = \sum_{j=0}^{c} \binom{c}{j} a^{c-j}\, b^j$.

(c) Show that a correct form of the position representation of the time evolution operator for non-interacting one-dimensional particles of mass $m = 1$ in a position-dependent potential $V(x)$ for an infinitesimal time step $\epsilon > 0$ is given by

$$U(x, t+\epsilon; y, t) = \langle x|\hat{U}(t+\epsilon, t)|y\rangle = \frac{1}{\sqrt{2\pi\mathrm{i}\epsilon}}\, \mathrm{e}^{\mathrm{i}(x-y)^2/(2\epsilon) - \mathrm{i}\epsilon V(y)} \tag{1}$$

(in units where $\hbar = 1$). To this end, you can start from the time evolution

$$\Psi(x, t+\epsilon) = \langle x|\Psi(t+\epsilon)\rangle = \langle x|\hat{U}(t+\epsilon, t)|\Psi(t)\rangle,$$

which connects the wave function $\Psi(x, t+\epsilon)$ with $\Psi(x, t)$. Then use $1 = \int \mathrm{d}y\, |y\rangle\langle y|$ and Eq. (1). Furthermore, express all functions of y except $\mathrm{e}^{\mathrm{i}(x-y)^2/(2\epsilon)}$ as Taylor series around x, i.e., in powers of $(y - x)$. After evaluating the integrals

$$\frac{1}{\sqrt{2\pi\mathrm{i}}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}az^2/2}\, \mathrm{d}z = \sqrt{\frac{1}{a}}, \tag{2}$$

$$\frac{1}{\sqrt{2\pi\mathrm{i}}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}az^2/2}\, z^2\, \mathrm{d}z = -2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}a} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\mathrm{i}}}\, \mathrm{e}^{\mathrm{i}az^2/2}\, \mathrm{d}z \tag{3}$$

(with $a > 0$), you can demonstrate, by identifying the Hamiltonian, that Eq. (1) indeed yields the correct time evolution for an infinitesimal time step.
Hint: During the calculation, take into account the result of part (b).

(d) Check whether, apart from $\lambda = 0$, there are further values of λ for which

$$U_\lambda(x, t+\epsilon; y, t) = \frac{1}{\sqrt{2\pi\mathrm{i}\epsilon}}\, \mathrm{e}^{\mathrm{i}(x-y)^2/(2\epsilon) - \mathrm{i}\epsilon(1-\lambda)V(y) - \mathrm{i}\epsilon\lambda V(x)} \tag{4}$$

is a correct form of the time evolution operator. How do you explain your result?

2. The path integral of particles in a magnetic field

In the following, we determine the path integral of non-interacting particles with charge e in a magnetic field in three spatial dimensions. In SI units, they are described by the Hamiltonian

$$\hat{H} = \frac{\left(\hat{\vec{p}} - e\vec{A}(\hat{\vec{r}})\right)^2}{2m}. \tag{5}$$

(a) Using the fundamental commutation relations, bring $\hat{H}$ into normal-ordered form, such that all momentum operators stand to the left of the position operators.

(b) Starting from part (a), show that the time evolution operator for an infinitesimal time step ϵ is given by

$$U(\vec{r}, t+\epsilon; \vec{r}\,', t) = \left(\sqrt{\frac{m}{2\pi \mathrm{i}\epsilon}}\right)^3 \mathrm{e}^{\mathrm{i}m(\vec{r}-\vec{r}\,')^2/(2\epsilon)}\, \mathrm{e}^{\mathrm{i}e\vec{A}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}\, \mathrm{e}^{-\epsilon\, e\vec{\nabla}\cdot\vec{A}(\vec{r}\,')/(2m)} \tag{6}$$

(with $\hbar = 1$).

(c) When constructing the path integral, it is not strictly necessary to normal-order the operators, as long as their non-commutativity is consistently taken into account. Show, by suitably splitting the Hamiltonian, that the time evolution can alternatively be described by the more symmetric expression

$$\widetilde{U}(\vec{r}, t+\epsilon; \vec{r}\,', t) = \left(\sqrt{\frac{m}{2\pi \mathrm{i}\epsilon}}\right)^3 \mathrm{e}^{\mathrm{i}m(\vec{r}-\vec{r}\,')^2/(2\epsilon)}\, \mathrm{e}^{\mathrm{i}e\frac{\vec{A}(\vec{r})+\vec{A}(\vec{r}\,')}{2}\cdot(\vec{r}-\vec{r}\,')}\, \mathrm{e}^{-\mathrm{i}\epsilon\, e^2(\vec{A}(\vec{r})-\vec{A}(\vec{r}\,'))^2/(8m)}. \tag{7}$$

3. Imaginary-time propagation

Compute the position representation of the time evolution operator $\langle x|\hat{U}(\tau)|x'\rangle = \langle x|\exp(-\hat{H}\tau)|x'\rangle$ for an arbitrary Hamiltonian with eigenstates $|n\rangle$ and eigenenergies E_n (here, τ denotes the imaginary time). By comparison with the explicit result

$$\langle x|\hat{U}(\tau)|x'\rangle = \sqrt{\frac{m\omega}{2\pi\sinh(\omega\tau)}}\exp\left\{-\frac{m\omega\left[(x^2+x'^2)\cosh(\omega\tau) - 2xx'\right]}{2\sinh(\omega\tau)}\right\},$$

determine the energy and the wave function of the ground state of the harmonic oscillator. Here, ω and m denote the eigenfrequency and the mass of the oscillator (with $\hbar = 1$).