Technische Universität Dresden Institut für Theoretische Physik

Exercises for "Theoretical Physics (Master)" Winter 26/27

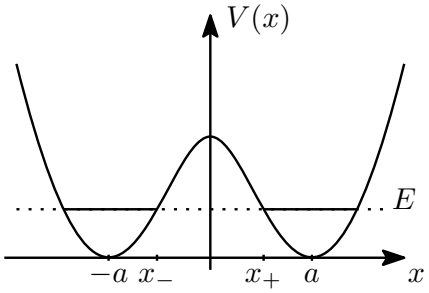
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Exercise 3 (17.-26.11.26)

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1. Tunneling and delocalized states

A particle in a double-well potential is described by the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, where $\hat{p}$ is the momentum operator canonically conjugate to the position operator $\hat{x}$, m denotes the mass of the particle, and the potential V has the form shown in the figure, with minima at $x = \pm a$ and $V(\pm a) = 0$.



(a) In the WKB approximation, one assumes that the potential V varies slowly on the scale of the de Broglie wavelength of the particle, such that the problem can be treated semiclassically. In this spirit, make the ansatz $\Psi(x) = A\,\mathrm{e}^{\mathrm{i}S(x)/\hbar}$ for the particle's wave function, formally expand $S(x)$ in powers of $\hbar$, and solve the Schrödinger equation to zeroth order in $\hbar$. Argue that the transmission probability $|T(E)|^2$ of a particle with energy E from the potential minimum at $x = -a$ to the minimum at $x = +a$ is of the form

$$|T(E)|^2 \sim \exp\left(-\frac{2}{\hbar}\int_{x_-(E)}^{x_+(E)} \mathrm{d}x\,\sqrt{2m\,[V(x)-E]}\right),$$

where $x_\pm(E)$ are the classical turning points at the barrier, $V(x_\pm) = E$ with $-a < x_- < 0 < x_+ < a$ (see figure). For $E = 0$, these are $x_\pm = \pm a$.

(b) In the path-integral formalism, the transition amplitude from $x = -a$ to $x = a$ during an imaginary time τ is given by

$$\langle a|\hat{U}(\tau)|-a\rangle = \int_{x(0)=-a,\,x(\tau)=a} \mathcal{D}x\,\exp\left\{-\frac{1}{\hbar}\int_0^\tau \mathrm{d}\tau'\left(\frac{m}{2}\dot{x}^2 + V(x)\right)\right\}.$$

The saddle point of this path integral corresponds to its classical solution. Show that the latter is determined by $m\dot{x}^2/2 = V(x)$ if the particle initially rests in the potential minimum at $x = -a$.

(c) Show that the simplest semiclassical approximation of the path integral reproduces the transmission probability $|T(0)|^2$ from part (a). Furthermore, approximate the potential near the right minimum as $V(a-\delta) = m\omega^2\delta^2/2$ (in imaginary time, this point is a maximum of the inverted potential $-V$), solve the equation of motion for small δ, and thereby show that the duration of the tunneling process is set by ω^{-1}.

(d) The next more complicated class of paths contributing to the transition amplitude is given by the particle tunneling back and forth several times. Describe these paths with the ansatz

$$\langle a|\hat{U}(\tau)|-a\rangle = \sum_{n\ \text{odd}} K^n \int_0^\tau \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2 \cdots \int_0^{\tau_{n-1}} \mathrm{d}\tau_n\, A_n(\tau_1,\ldots,\tau_n),$$

where $A_n(\tau_1, \ldots, \tau_n)$ describes the contribution of a path in which the particle tunnels a total of n times between the minima (these tunneling events take place at the times τ_i), and K parametrizes the prefactor. Decompose $A_n = A_{n,\,\mathrm{sc}}\, A_{n,\,\mathrm{qm}}$ into a semiclassical contribution $A_{n,\,\mathrm{sc}}$, which describes the tunneling events analogously to parts (b) and (c), and a quantum-mechanical contribution $A_{n,\,\mathrm{qm}}$, which corresponds to the particle residing in the potential minima between the tunneling events. To this end, assume that between the tunneling events the particle is approximately in the ground state of the harmonic-oscillator potential $V(\pm a - \delta) = m\omega^2\delta^2/2$, while the tunneling itself is quasi-instantaneous.

(e) Physically, the effect of tunneling can be described by symmetric and antisymmetric states that are delocalized between the two potential minima. If the potential near the minima is approximated as harmonic, the two ground states, which are degenerate in the absence of tunneling, acquire energies $E = \hbar\omega/2 \pm \Delta E/2$. Use the identity

$$\sum_{n\ \mathrm{odd}} \frac{1}{n!} \left(A\,\mathrm{e}^{-B}\right)^n = \sinh\!\left(A\,\mathrm{e}^{-B}\right)$$

to determine ΔE from the result obtained in part (d). For this purpose, consider $\langle a|\hat{U}(\tau)|-a\rangle$ in the approximation that only the symmetric and antisymmetric superpositions of the (former) ground states contribute to the time evolution.

2. Landau levels in the symmetric gauge

Determine the wave functions of a free electron in a homogeneous magnetic field $\mathbf{B} = (0, 0, B)^\top$ in the symmetric gauge $\mathbf{A} = (-By/2, Bx/2, 0)^\top$.

(a) To this end, introduce the independent complex coordinates $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$. What is the explicit form of the derivative operators ∂_z and ∂_{z^*} in terms of derivatives with respect to x and y? Justify that the vector potential has the components $A_z = (A_x - \mathrm{i}A_y)/2$ and $A_{z^*} = (A_x + \mathrm{i}A_y)/2$.

Note: In this exercise, z always denotes the complex coordinate, and A_z, A_{z^*} the corresponding components of $\mathbf{A}$, not Cartesian components along the field direction.

(b) Now consider the Hamiltonian $\hat{H}$ in the symmetric gauge and in the new coordinates z and z^*. First restrict yourself to the case $eB > 0$ (where e is the charge of the electron), and define $\hat{H}'$ through

$$\hat{H} = \mathrm{e}^{-zz^*/(4a_B^2)}\, \hat{H}'\, \mathrm{e}^{zz^*/(4a_B^2)},$$

where $a_B = \sqrt{\hbar/|eB|}$ is the magnetic length, $\omega_c = |eB|/m$ denotes the cyclotron frequency, and m is the electron mass. Show that $\hat{H}'$ can be written in the form

$$\hat{H}' = -\frac{\hbar^2}{2m}\left(4\partial_z - \frac{2eB}{\hbar}\, z^*\right)\partial_{z^*} + \frac{\hbar\omega_c}{2}.$$

What changes for $eB < 0$?

Hint: For simplicity, you may restrict yourself to the sector of the Hilbert space with vanishing momentum along the magnetic field, $\hat{\mathbf{p}} \cdot \mathbf{B} = 0$. (Why?)

(c) What is the general form of the wave functions of the degenerate states in the lowest Landau level? Compare your results with those of a calculation in the Landau gauge.