Ultracold Quantum Gases

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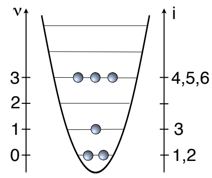
Problem set 3 to be discussed at 13:00h on 9th of June, 2011

. see http://tu-dresden.de/physik/tqo/lehre

Ideal $1d$-Bose-Gas in a harmonic trap (canonical ensemble) I

We denote with $\{n\}_N$ a set of occupation numbers n_ν with $\sum_{\nu=0}^{\infty} n_\nu = N$ (see the figure: $n_0 = 2, n_1 = 1, n_2 = 0, n_3 = 3, N = 6$). Then the canonical partition function Z_N of N non-interacting Bosons in a one-dimensional harmonic trap (frequency ω) reads $Z_N = \sum_{\{n\}_N} \exp\left(-\beta E_{\{n\}_N}\right)$, where $E_{\{n\}_N} = \hbar\omega \sum_{\nu=0}^{\infty} n_\nu \nu$. In order to evaluate the partition function we choose a new way of summation: starting with the ground state, we assign an index i to each atom and denote with ν_i the corresponding energy level. For the situation in the figure, $\nu_1 = 0, \nu_2 = 0, \nu_3 = 1, \nu_4 = 3, \nu_5 = 3, \nu_6 = 3$.



a) Show that $Z_N = \sum_{(\nu)_<} \exp\left(-\beta\hbar\omega \sum_{i=1}^{N} \nu_i\right)$, where the summation is extended over all ordered N-tuples $(\nu)_< = (\nu_1, \ldots, \nu_N)$ with $0 \leq \nu_1 \leq \nu_2 \leq \ldots \leq \nu_N < \infty$.

b) Use a) to find $Z_N = \Pi_{i=1}^{N}(1-q^i)^{-1}$ with $q = e^{-\beta\hbar\omega}$.

c) Determine average energy $\langle E \rangle$ and heat capacity $C = \frac{\partial \langle E \rangle}{\partial T}$ of the gas.

d) What is the limiting behaviour of $\langle E \rangle$ and C for large temperatures $\frac{k_B T}{\hbar\omega} \gg 1$?

Ideal $1d$-Bose-Gas in a harmonic trap (canonical ensemble) II

Here we determine the full distribution $P(N_0)$ of the ground state occupation number (same notations as above).

a) Confirm that $P(N_0) = \frac{1}{Z_N} \sum_{\{n'\}_{N-N_0}} \exp\left(-\beta\hbar\omega E_{\{n'\}_{N-N_0}}\right)$ where the sum $\{n'\}$ runs over all occupation numbers of *excited* modes ($\nu > 0$) with constraint $\sum_{\nu=1}^{\infty} n_\nu = N - N_0$.

b) Show that the probability $P_>(N_0)$ to find *at least* N_0 atoms in the ground state is $P_>(N_0) = \frac{Z_{N-N_0}}{Z_N}$. With the help of the results of the first problem find $P(N_0) = q^{N-N_0} \Pi_{i=1}^{N_0} \left(1 - q^{N-N_0+i}\right)$ (with $q = e^{-\beta\hbar\omega}$).

c) Now consider low temperatures $\beta\hbar\omega(N - N_0) \gg 1$, and confirm:
$\langle N_0 \rangle = N - \left(\frac{k_B T}{\hbar\omega}\right) \ln(e^{\beta\hbar\omega} - 1)$; $\langle (\Delta N_0)^2 \rangle = \left(\frac{k_b T}{\hbar\omega}\right)^2$.

d) Compare to the corresponding expressions for the grand-canonical ensemble.

Please turn over !!!

Bose-Einstein condensation in a $d = 3$ dimensional box $V = L^3$

Below T_c the ground state with energy $\epsilon_0 = 0$ is macroscopically populated and requires special treatment, for instance $\langle N \rangle = \langle N_0 \rangle + \sum_{\vec{k} \neq 0} \langle n_{\vec{k}} \rangle$. This raises the question whether further low-lying states need special attention, too.

a) Determine the energy ϵ_1 of the first excited state. How many such states are there?

b) How does ϵ_1 scale with volume V and thus in the thermodynamic limit ($\frac{\langle N \rangle}{V}$ =const) with particle number $\langle N \rangle$?

c) Determine the occupation number $\langle n_1 \rangle$ for large $\langle N \rangle$ and confirm that indeed, $\langle n_1 \rangle$ may be neglected with respect to $\langle N_0 \rangle$ in that limit.

Bose gas in $d = 2$ dimensional box $V = L^2$

As shown in the lectures, for a Bose gas in a d-dimensional box and for given $\langle N \rangle$, fugacity $z = e^{\beta \mu}$ is fixed by the equation

$$\frac{\Lambda^d \langle N \rangle}{V} = \frac{\Lambda^d}{V} \frac{z}{1-z} + g_{d/2}(z).$$

Here $\Lambda = \sqrt{2\pi\hbar^2/(mk_BT)}$ is the thermal de Broglie-wave length and $g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$ the Bose function introduced in the lectures.

a) Show: there is *no* Bose-Einstein condensation in a box in $d = 2$ dimensions, i.e. $\frac{\langle N_0 \rangle}{V} \to 0$ in the thermodynamic limit.

b) Determine (for $d = 2$) the ground state occupation number $\langle N_0 \rangle$ (in the thermodynamic limit) for low temperatures $\Lambda^2 \frac{\langle N \rangle}{V} \gg 1$.