Ultracold Quantum Gases

Walter Strunz, Institut für Theoretische Physik , TU Dresden, Sommersemester 2011

Problem set 4

to be discussed at 13:00h on 23th of June, 2011

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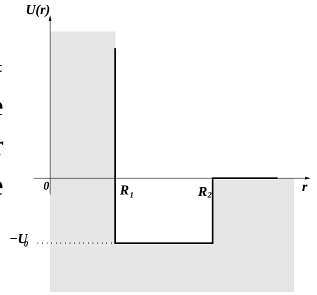
see http://tu-dresden.de/physik/tqo/lehre

Scattering length and resonances



For the potential shown in the figure ($U(r) = \infty$ for $0 < r < R_1$; $U(r) = -U_0 = -\hbar^2\kappa^2/2m$ for $R_1 < r < R_2$; and $U(r) = 0$ for $r > R_2$) determine the s-wave-scattering length a. (Hint: first calculate the general expression for $\tan\delta_0(k)$ and then consider $k \to 0$). What can be said about the discrete spectrum ($E < 0$)? Consider, in particular, the limit $E \to 0^-$.

Interacting uniform Bose-Gas

Determine the Hamiltonian $\hat{H}$ of an interacting Bose Gas with two-particle interaction potential $U(\vec{r}_1 - \vec{r}_2)$ in a box of volume $V = L^3$ in second quantization. Use $\hat{\psi}(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} \hat{a}_{\vec{k}} e^{i\vec{k}\vec{r}}$ and express $\hat{H}$ in terms of momentum space operators $\hat{a}_{\vec{k}}$. Finally, consider the case $U(\vec{r}) = U_0 \cdot \delta(\vec{r})$.

Virial theorem for Gross-Pitaevskii equation

Let $\psi_0(\mathbf{r})$ be the solution of the time-independent Gross-Pitaevskii equation

$$\left(-\frac{\hbar^2}{2m}\Delta + V(\mathbf{r}) + g|\psi_0(\mathbf{r})|^2\right)\psi_0(\mathbf{r}) = \mu\psi_0(\mathbf{r})$$

with a harmonic trap $V(\mathbf{r}) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$ and normalization $\int d\mathbf{r}|\psi_0(\mathbf{r})|^2 = N$. Total energy E_{GP} is a sum of kinetic, potential, and internal interaction energy: $E_{GP} = E_{kin} + E_{pot} + E_{int}$. Show that:

a) $E_{kin} + E_{pot} + 2E_{int} = N\mu$.

b) $(E_{kin})_x - (E_{pot})_x + \frac{1}{2}E_{int} = 0$; (and similar for y and z).

[Hint: ψ_0 is a solution of the variational principle $\delta E_{GP}[\psi] = 0$; consider variations of the kind $\psi(x, y, z) \to \sqrt{1+\epsilon}\,\psi_0\,((1+\epsilon)x, y, z)$.]

c) $2E_{kin} - 2E_{pot} + 3E_{int} = 0$.

Please turn over !!!

Bose-Einstein condensate with attractive interaction: limited condensate number

We consider a Bose-Einstein condensate (particle number N) with *attractive* interaction ($g = \frac{4\pi\hbar^2 a}{m} < 0$) in an isotropic harmonic trap: $U(\mathbf{r}) = \frac{1}{2}m\omega^2\mathbf{r}^2$.

a) Assuming the condensate (product) form of the N-particle wave function $\Psi_C(\vec{r}_1, \ldots, \vec{r}_N) = \phi(\vec{r}_1)\cdots\phi(\vec{r}_N)$, determine the energy $E[\phi, \phi^*] = \langle H \rangle$ of the N-particle system assuming a *Gaussian* single-particle wave function $\phi(\mathbf{r}) = (\pi R^2)^{-\frac{3}{4}} \exp\{-\frac{\mathbf{r}^2}{2R^2}\}$ of extension R.

b) Sketch the condensate energy $E = E(R) = \langle H \rangle$ as a function of the size R.

c) Show that for a limited number of atoms N, there is a local minimum of $E(R)$.

d) Determine the critical number N_c above which this local minimum disappears and interpret your result.

e) Read *Bradley, Sackett, Hulet; Phys. Rev. Lett.* **78**, 985 (1997).