
Problem set 5

to be discussed at 13:00h on 7th of July, 2011

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see http://tu-dresden.de/physik/tqo/lehre

"Black Holes" in Bose-Einstein condensates

In the hydrodynamic limit the solution of the time dependent Gross-Pitaevskii equation with external potential $V(\mathbf{x})$ is written as $\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\phi(\mathbf{x},t)}$. Then one derives equations for ρ and ϕ, neglecting gradients of ρ.

a) Linearise these equations around a stationary solution $\psi_s(\mathbf{x},t) = \sqrt{\rho_s(\mathbf{x})}e^{i\phi_s(\mathbf{x})-i\mu t}$ of the GP equation; i.e. write $\rho = \rho_s + \delta\rho$ and $\phi = \phi_s + \delta\phi$ with small $\delta\rho$ and $\delta\phi$. Notice that in contrast to the lectures, we here allow for a velocity field $\mathbf{V}(\mathbf{x}) = \frac{\hbar}{m}\nabla\phi_s(\mathbf{x})$ different from zero.

b) Eliminate $\delta\rho$ to derive of a wave equation for $\delta\phi$. Show that it can be written in the form $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\phi) = 0$ ($\mu,\nu = 0,\ldots,3$ and $\partial_0 = \partial_t, \partial_1 = \partial_x, \ldots$, summation convention). Here, $g_{\mu\nu}$ is the $(4\times 4)-$ metric tensor

$$(g_{\mu\nu}) = c\begin{pmatrix} -(c^2-\mathbf{V}^2) & -\mathbf{V}^T \\ -\mathbf{V} & \mathbf{1} \end{pmatrix},$$

with $c = c(\mathbf{x}) = \frac{\hbar}{m}\sqrt{4\pi a\rho_s(\mathbf{x})}$ the local speed of sound and $\mathbf{V} = \mathbf{V}(\mathbf{x}) = \frac{\hbar}{m}\nabla\phi_s(\mathbf{x})$ the local velocity field.

c) Interpret the above result for the dynamics of phase fluctuations $\delta\phi$. Draw an analogy to black hole physics.

d) See Garay, Anglin, Cirac, Zoller: Phys. Rev. Lett. **85**, 6443 (2000) and Phys. Rev. A **63**, 023611 (2001) (find these articles on the materials web page).

Please turn over!

Bogoljubov transformation

In order to discuss excitations from a condensate of an interacting homogenous gas ($V = L^3$), we write $\hat{\psi}(\vec{r}) = \psi_0 + \hat{\phi}(\vec{r})$, with the condensate $\psi_0 = \sqrt{n_0}$ a real number and n_0 the condensate density. The total energy is expanded in powers of $\hat{\phi}$. Choose momentum representation and show that the second order contribution H_2 with the usual interaction parameter $g = \frac{4\pi\hbar^2 a}{m}$ reads

$$H_2 = \sum_{\vec{k}} {}' \left\{ (\frac{\hbar^2 k^2}{2m} + gn_0) a^+_{\vec{k}} a_{\vec{k}} + \frac{gn_0}{2} (a_{\vec{k}} a_{-\vec{k}} + a^+_{\vec{k}} a^+_{-\vec{k}}) \right\}.$$

Define new Bosonic quasi-particle creation and annihilation operators $\alpha_{\vec{k}}, \alpha^+_{\vec{k}}$ through the ansatz $a_{\vec{k}} = u_{\vec{k}} \alpha_{\vec{k}} - v_{\vec{k}} \alpha^+_{-\vec{k}}$. Determine the coefficients $u_{\vec{k}}, v_{\vec{k}}$ such that H_2 becomes diagonal $H_2 = \sum_{\vec{k}} {}' E_{\vec{k}} \alpha^+_{\vec{k}} \alpha_{\vec{k}} + \Delta C$ and read off the quasi-particle excitation energies $E_{\vec{k}}$.

Attractive interaction II

Discuss the quasiparticle excitation spectrum above for the case of attractive interaction $a < 0$. Consider finite values of N_0 and V, and also the thermodynamic limit.