Problem set 4

to be discussed on Thursday, December 10th, 2009

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see http://tu-dresden.de/physik/tqo/lehre

Linear first order partial differential equation

Let $\rho(\vec{x},t)$ be the solution of

$$\frac{\partial}{\partial t}\rho = -\sum_i \frac{\partial}{\partial x_i}\left(v_i(\vec{x},t)\rho\right)$$

with initial condition $\rho(\vec{x},t=0)=\rho_0(\vec{x})$. Show that

$$\rho(\vec{x},t) = \int d\vec{x}_0\, \rho_0(\vec{x}_0)\, \delta(\vec{x}-\vec{x}(t))$$

where $\vec{x}(t)$ is the trajectory of the dynamical system $\dot{\vec{x}} = \vec{v}$ with initial condition $\vec{x}(t=0)=\vec{x}_0$.
Give a descriptive interpretation of the result.
Consider the special case of a Hamiltonian dynamical system and state the corresponding evolution equation for a phase space density in its well-known form.

Ornstein-Uhlenbeck process I

$Y(t)$ being the (standard) Ornstein-Uhlenbeck process define $Z(t)=\int_0^t Y(t')dt'$.
Is $Z(t)$ Gaussian? Is $Z(t)$ stationary? Is $Z(t)$ Markovian? Show that

$$\langle Z(t_1)Z(t_2)\rangle = e^{-t_1}+e^{-t_2}-1-e^{-|t_1-t_2|}+2\text{min}(t_1,t_2).$$

Ornstein-Uhlenbeck process II

For the same $Z(t)$ find the characteristic functional and use it to obtain

$$\langle\cos\left(Z(t_1)-Z(t_2)\right)\rangle = \exp\left(-e^{-|t_1-t_2|}+1+|t_1-t_2|\right).$$

Ornstein-Uhlenbeck process III

In the Ornstein-Uhlenbeck process rescale the variables: $y=\alpha y'$, $t=\beta t'$ and show that in a suitably chosen limit of α and β the $P_{1|1}$ reduces to that of the Wiener process.