Quantum Stochastic Processes

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Problem set 5

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see http://tu-dresden.de/physik/tqo/lehre

Stochastic integral

Complete the proof sketched in the lectures:
With $W(t)$ the Wiener process, use the definition of Ito's stochastic integral to show that in the mean square limit

$$\int_{t_0}^{t} W(s)\mathrm{d}W(s) = \frac{1}{2}\left(W(t)^2 - W(t_0)^2\right) - \frac{1}{2}(t-t_0).$$

Wiener and Ornstein-Uhlenbeck process

Starting from the known diffusion equations for Wiener and Ornstein-Uhlenbeck process, determine the corresponding Ito-stochastic differential equations. In which sense can we relate both equations to Langevin's original equation "$m\ddot{x}(t) + 6\pi\eta a\dot{x}(t) = X(t)$"?

Complex Wiener increments

Let $\mathrm{dW_1(t)}$ and $\mathrm{dW_2(t)}$ be increments of two independent Wiener processes $W_1(t)$ and $W_2(t)$. State Ito-rules for complex increments

$$\mathrm{d}\xi \equiv \frac{1}{\sqrt{2}}\left(\mathrm{dW_1} + \mathrm{i}\,\mathrm{dW_2}\right).$$

Linear Ito-stochastic Schrödinger equation

Consider an Ito-stochastic extension of the usual Schrödinger equation of the form

$$\mathrm{d}|\psi\rangle = \frac{1}{\mathrm{i}\hbar}H|\psi\rangle\mathrm{d}t - \frac{1}{2}L^\dagger L|\psi\rangle\mathrm{d}t + L|\psi\rangle\mathrm{d}\xi.$$

Here H is the Hamiltonian and L an operator (not necessarily self-adjoint) modelling the influence of some "environment". Derive an evolution equation for the density operator

$$\rho(t) = \langle\!\langle\ |\psi(t)\rangle\langle\psi(t)|\ \rangle\!\rangle,$$

where the average $\langle\!\langle \ldots \rangle\!\rangle$ denotes an ensemble average over the complex Wiener process $\xi(t)$ (see problem above). Is the trace of $\rho(t)$ preserved? What can be said about the norm $\langle\psi(t)|\psi(t)\rangle$ of a single realization?

Non-linear Ito-stochastic Schrödinger equation

Same as above for the non-linear Ito-stochastic Schrödinger equation

$$\mathrm{d}|\psi\rangle = \frac{1}{\mathrm{i}\hbar}H|\psi\rangle\mathrm{d}t - \frac{1}{2}\left(L^\dagger L - 2\langle L^\dagger\rangle L + |\langle L\rangle|^2\right)|\psi\rangle\mathrm{d}t + \left(L - \langle L\rangle\right)|\psi\rangle\mathrm{d}\xi,$$

where $\langle L\rangle = \langle\psi(t)|L|\psi(t)\rangle$.