Tutorial on Critical Phenomena, Scaling and the Renormalization Group

Problem 1: An exact solution of the one-dimensional Ising model

The exact calculations of free energy and correlation functions of the one-dimensional Ising model demonstrate, that there is a critical point at zero temperature at which the correlation length and susceptibility diverge.

In the one-dimensional Ising model, there is a spin variable $s_i = \pm 1$ at each site $i = 1,2, \dots, N$ on a one-dimensional lattice. Assuming only nearest neighbor interactions the Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i \ ,$$

where $\langle i,j \rangle$ denotes the sum over neighboring lattice sites. J and h are the exchange integral ($J > 0$ for ferromagnetic interaction) and the external magnetic field, respectively.

(1) The partition function of this model

$$\mathcal{Z}_N(T,h) = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \dots \sum_{s_N=\pm 1} e^{-\frac{\mathcal{H}}{k_B T}} \equiv \sum_{s_1,s_2,\dots,s_N} e^{-\frac{\mathcal{H}}{k_B T}}$$

can be calculated exactly for arbitrary h using the formalism of *transfer matrices.* By using periodic boundary conditions (in which s_{N+1} is identical to s_1), show first that the partition function can be expressed as a trace of a product of transfer matrices $\underline{T}$

$$\mathcal{Z}_N = \mathrm{Tr}\, \underline{T}^N \ ,$$

where $T_{s_i s_{i+1}} = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h} \end{pmatrix} \equiv \underline{T}$ is a two-by-two matrix (with $\beta \equiv 1/k_B T$).

(2) Calculate the eigenvalues $\lambda_\pm$ of the matrix $\underline{T}$ and show that in the thermodynamic limit the Gibbs *free energy per spin* becomes (for $T \neq 0$)

$$g(T,h) := \lim_{N\to\infty} \frac{1}{N} \left(-k_B T \ln \mathcal{Z}_N(T,h)\right) = \lim_{N\to\infty} \frac{1}{N} \left(-k_B T \ln(\lambda_+^N + \lambda_-^N)\right) = -\, k_B T \ln \lambda_+$$

with $\lambda_+ = e^{\beta J} \left\{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}\right\}$.

What follows in the low T low $\tilde{h}$ limit (with $\tilde{h} = \beta h$)?

(3) Calculate the (dimensionless) equilibrium magnetization per spin $m(T,h) := -\left(\frac{\partial g}{\partial h}\right)_T$ in the thermodynamic limit and show that for the one-dimensional Ising model the spontaneous magnetization $m(T,0)$ vanishes for all temperatures $T \neq 0$.

The susceptibility measuring the change in $m(T,h)$ in response to an external field is $\chi := \left(\frac{\partial m}{\partial h}\right)_T$. Determine χ in the limit $h \to 0$.

(4) The spin-spin correlation function $G(na)$ with two spins s_i, s_{i+n} separated by a distance na (a denotes the lattice constant) is defined by

$$G(na) := \langle [s_i - \langle s_i \rangle][s_{i+n} - \langle s_{i+n} \rangle] \rangle \equiv \langle \delta s_i \, \delta s_{i+n} \rangle \ ,$$

where $\langle \mathcal{A} \rangle := \frac{1}{\mathcal{Z}_N} \sum_{s_1,s_2,\dots s_N} \mathcal{A} e^{-\beta \mathcal{H}}$.

Show that for $h = 0$ we have for the correlation function in the limit $N \to \infty$:

$$G(na) = \tanh^n(\beta J) \ .$$

Hint:

If $h = 0$, the matrix element $T_{s_i s_{i+1}}$ takes on the simple form $T_{s_i s_{i+1}} = \cosh(\beta J)\ [1 + s_i s_{i+1} \tanh(\beta J)]$ and we easily obtain $\left(\underline{T}^2\right)_{s_i s_{i+2}} = \sum_{s_{i+1}} T_{s_i s_{i+1}} T_{s_{i+1} s_{i+2}} = 2\cosh^2(\beta J)\ [1 + s_i s_{i+2} \tanh^2(\beta J)]$. From this we can deduce $\left(\underline{T}^l\right)_{s_i s_{i+l}}$.

(5) Use the result of (4) and the relation $G(r) \sim e^{-r/\xi}$ (valid for large r) to determine the correlation length ξ. What follows in the high- and low-temperature limits?

Problem 2: Renormalization group for the one-dimensional Ising model

Using the Migdal-Kadanoff procedure we apply the Renormalization Group (RG) technique to a one-dimensional Ising model (where the RG is exact). The Migdal-Kadanoff procedure is one of the easiest concepts of RG applicable to lattice spin Hamiltonians. It is based on the elimination of a certain fraction of spins from the partition sum, reducing in this way the number of degrees of freedom. The removed spins induce an effective interaction of the remaining spins which causes renormalized coefficients in the Hamiltonian of the new reduced spin system. This decimation process is continued indefinitely. From the resulting RG flows the fixed points of the RG transformations are deduced. The exponents obtained by linearizing the recursion relations in the vicinity of unstable critical fixed points (unstable at least in one direction) are the exponents determining the scaling of the free energy and of related functions. From the scaling behavior in turn we can deduce the critical exponents.

Consider the one-dimensional Ising model with nearest neighbor interactions. The reduced Hamiltonian is $\tilde{\mathcal{H}} = -\frac{\mathcal{H}}{k_B T} = \sum_{i=1}\left\{\tilde{J} s_i s_{i+1} + \frac{1}{2}\tilde{h}(s_i + s_{i+1})\right\}$, where $\tilde{J} = \frac{J}{k_B T}$ and $\tilde{h} = \frac{h}{k_B T}$.

(1) Decimate the number of degrees of freedom by summing in the partition sum over all even numbered spins (without loss of generality we assume that the total number of spins is even). Show that the partition function $\mathcal{Z}_N(\tilde{J},\tilde{h}) \equiv \mathcal{Z}_N[\tilde{\mathcal{H}}]$ can be expressed by the new (renormalized) Hamiltonian $\tilde{\mathcal{H}}'$ involving spins only at odd numbered sites

$$\mathcal{Z}_N[\tilde{\mathcal{H}}] = \sum_{\sigma_1,\sigma_2,\sigma_3,\ldots} e^{\tilde{\mathcal{H}}} = \sum_{\sigma_1,\sigma_3,\ldots} e^{\tilde{\mathcal{H}}'} = \mathcal{Z}_{N'=N/2}[\tilde{\mathcal{H}}'] \ ,$$

where $\tilde{\mathcal{H}}'$ has a similar form as the original Hamiltonian $\tilde{\mathcal{H}}$ but with renormalized coupling parameters $\tilde{J}', \tilde{h}'$ determined by

$$\left(e^{\tilde{J}(s_1+s_3)+\tilde{h}} + e^{-\tilde{J}(s_1+s_3)-\tilde{h}}\right)e^{\frac{1}{2}\tilde{h}(s_1+s_3)} = \phi(\tilde{J},\tilde{h})\, e^{\tilde{J}' s_1 s_3 + \frac{1}{2}\tilde{h}'(s_1+s_3)} \ .$$

This means that the remaining spins of the thinned-out lattice interact with their nearest neighbors through a renormalized coupling parameter $\tilde{J}'$ and are subject to a renormalized external field $\tilde{h}'$.

(2) By substituting the different values for s_1, s_3 obtain the relationship between the renormalized parameters $\tilde{J}', \tilde{h}'$ and the original parameters $\tilde{J}, \tilde{h}$.

(3) The procedure described in (2) may be repeated again and again. To understand the resulting "flow" of the parameters $\tilde{J}, \tilde{h}$ we first consider the case without external field.
Show that for $\tilde{h} = 0$ it follows

$$\tilde{J}' = \frac{1}{2}\ln\cosh(2\tilde{J}) \le \tilde{J} \ .$$

Demonstrate that this implies a stable fixed point at $\tilde{J} = 0$ (infinite temperature T) and an unstable fixed point at $\tilde{J} = \infty$ (zero T) of the RG transformation.
Starting from any finite interaction $\tilde{J}$ the successive thinning out of degrees of freedom leads to a Hamiltonian where the remaining spins are more weakly coupled. This indicates that the one-dimensional Ising chain is at any temperature $T \neq 0$ always disordered at sufficiently long length scales.

(4) The relationship between $\tilde{J}'$ and $\tilde{J}$ of (3) can also be written as

$$\tanh(\tilde{J}') = \tanh^2(\tilde{J}) \ .$$

Use this equation to find a recursion relation for the correlation length ξ measured in terms of the lattice constant of the respective lattice.
Show that the (dimensionless) correlation length decreases under the renormalization procedure unless the system is critical ($\xi = \infty$) or noninteracting ($\xi = 0$).

Hint:
Remember that in Problem 1 we have shown that for the spin-spin correlation function of the one-dimensional Ising model it holds $g(na) = \tanh^n(\tilde{J})$].

(5) Now we return to the more general case of nonzero external field and are interested in the flow of coupling parameters for $\tilde{h} \neq 0$. Show that $\frac{\partial \tilde{h}\prime}{\partial \tilde{h}} > 1$ for all finite $\tilde{J}$.

(6) The recursion relations for $\tilde{J}$ and $\tilde{h}$ (see (2)) can be linearized around the unstable fixed point (the $T = 0$ critical point). Show that in the vicinity of this critical point

$$e^{-\tilde{J}'} = \sqrt{2}e^{-\tilde{J}} \quad , \quad \tilde{h}' = 2\,\tilde{h} \quad .$$

(7) Now regard $e^{-\tilde{J}}$ and $\tilde{h}$ as scaling fields. Show that then for the singular part of the free energy per spin g_{sing} it follows

$$\tilde{g}_{\text{sing}}\left(e^{-\tilde{J}}, \tilde{h}\right) = \lambda^{-1}\tilde{g}_{\text{sing}}\left(\lambda^{1/2}e^{-\tilde{J}\xi}, \lambda\tilde{h}\right) \qquad (\text{with } \tilde{g}_{\text{sing}} \equiv \frac{g_{\text{sing}}}{k_{\text{B}}T})$$

for rescaling parameter $\lambda = 2$.

Demonstrate that this implies the following scaling law for $\tilde{f}_{\text{sing}}$

$$\tilde{g}_{\text{sing}}\left(e^{-\tilde{J}}, \tilde{h}\right) = e^{-2\tilde{J}}\tilde{g}_{\text{sing}}\left(\tilde{h}e^{2\tilde{J}}\right) \ .$$

(8) Express the above scaling law in terms of the correlation length ξ rather than $\tilde{J}$. Show that the critical indices (exponents) γ, ν, α (describing the strength of leading singularities in susceptibility, correlation length and specific heat, respectively, near the critical point) satisfy the following relation

$$\frac{\gamma}{\nu} = \frac{2-\alpha}{\nu} = 1 \, .$$

For more details see, for example, P.M Chaikin and T.C: Lubensky, *Principles of condensed matter physics* (Cambridge University Press)