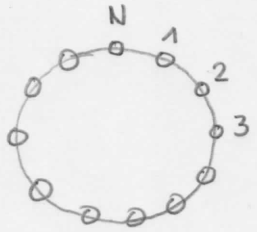
Solution of Problem 1

(1) Bulk properties in the thermodynamic limit should be insensitive to boundary conditions

$\rightsquigarrow$ calculations can be carried out using periodic boundary conditions

here: lattice sites lie on a circle with sites 1 and N connected



Partition function:

$$Z_N = \sum_{s_1}\sum_{s_2}\dots\sum_{s_N} e^{-\beta\mathcal{H}} = \sum_{s_1}\dots\sum_{s_N} e^{\beta J\sum_{i=1}^{N} s_i s_{i+1} + \beta h \sum_{i=1}^{N} s_i}$$

$$\beta = \frac{1}{k_B T}$$

$$= \sum_{s_1}\dots\sum_{s_N} \underbrace{e^{\beta J s_1 s_2 + \frac{\beta h}{2}(s_1+s_2)}}_{\equiv T_{s_1 s_2}} \underbrace{e^{\beta J s_2 s_3 + \frac{\beta h}{2}(s_2+s_3)}}_{\equiv T_{s_2 s_3}} \cdot \dots$$

$$\dots \cdot \underbrace{e^{\beta J s_N s_1 + \frac{\beta h}{2}(s_N+s_1)}}_{\equiv T_{s_N s_1}}$$

Every factor $T_{s_i s_{i+1}} := e^{\beta J s_i s_{i+1} + \frac{\beta h}{2}(s_i+s_{i+1})}$

can be understood as a matrix element of a two-by-two matrix $\underline{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} e^{\beta J+\beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta h} \end{pmatrix}$

$$\rightsquigarrow \quad Z_N = \sum_{s_1}\dots\sum_{s_N} T_{s_1 s_2} T_{s_2 s_3}\dots T_{s_N s_1} = \sum_{s_1} \left(\underline{T}^N\right)_{s_1 s_1}$$

$$= \mathrm{Tr}\left(\underline{T}^N\right)$$

(2) Now we diagonalize $\underline{T}$: $\underline{T} \to \underline{T}' = \underline{A}^{-1} \underline{T} \underline{A}$

$$\leadsto \quad \mathrm{Tr}\,(\underline{T}^N) = \mathrm{Tr}\,(\underline{T}')^N = \lambda_+^N + \lambda_-^N$$

λ_+, λ_- eigenvalues of $\underline{T}$

$$\begin{vmatrix} (e^{\beta J + \beta h} - \lambda) & e^{-\beta J} \\ e^{-\beta J} & (e^{\beta J - \beta h} - \lambda) \end{vmatrix} \stackrel{!}{=} 0$$

$$\leadsto \lambda_\pm = e^{\beta J} \left(\cosh(\beta h) \pm \sqrt{\sinh^2(\beta h) + e^{-4\beta J}} \right)$$

$$\leadsto \quad \lambda_+ > |\lambda_-| \quad \text{for arbitrary } h \quad (\text{if } T \neq 0)$$

$$\frac{1}{N} \ln Z_N = \frac{1}{N} \ln (\lambda_+^N + \lambda_-^N) = \ln \lambda_+ + \frac{1}{N} \ln\left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right)$$

$$\xrightarrow{N \to \infty} \ln \lambda_+$$

$\leadsto$ Gibbs free energy per spin

$$g(T,h) := \lim_{N \to \infty} \frac{1}{N} (-k_B T \ln Z)$$

$$= -J - k_B T \ln \left\{ \cosh\left(\frac{h}{k_B T}\right) + \sqrt{\sinh^2\left(\frac{h}{k_B T}\right) + e^{-4J/k_B T}} \right\}$$

In the limit low T, low $\tilde{h}$ (with $\tilde{h} \equiv \frac{h}{k_B T}$, $\frac{h}{k_B T} e^{2J/k_B T} \ll 1$)

$$g(T,h) = -J - k_B T e^{-2J/k_B T} - \frac{1}{2} \frac{h^2}{k_B T} e^{2J/k_B T}$$

(3) Magnetization per spin

$$m(T,h) := -\left(\frac{\partial g}{\partial h}\right)_T = \frac{\sinh(h/k_BT)}{\sqrt{\sinh^2(h/k_BT) + e^{-4J/k_BT}}}$$

$\leadsto$ $m(T,0) = 0$ $\leadsto$ spontaneous magnetization $= 0$ for all $T \neq 0$

$\leadsto$ one-dim. Ising model is for all $T \neq 0$ paramagnetic

Magnetization for low h: $m(T,h) \approx e^{2J/k_BT} \frac{h}{k_BT}$

$\leadsto$ Susceptibility for low h: $\chi = \left(\frac{\partial m}{\partial h}\right)_T = \frac{1}{k_BT} e^{2J/k_BT}$

$\leadsto$ χ diverges as $T \to 0$

$\leadsto$ indicating a critical point at $T=0$

(4) In the zero field case $\langle s_i \rangle = 0$

$\leadsto$ ~~spin-spin~~ correlation function

$$\begin{aligned}
G(na) &= \langle (s_i - \langle s_i \rangle)(s_{i+n} - \langle s_{i+n} \rangle) \rangle \\
&= \langle s_i s_{i+n} \rangle \\
&= \frac{1}{Z_N} \sum_{s_1} \dots \sum_{s_N} s_1 s_{1+n} e^{\beta J \sum_{i=1}^{N} s_i s_{i+1}} \\
&= \frac{\sum_{s_1} \sum_{s_{1+n}} s_1 s_{1+n} (\underline{T}^n)_{s_1 s_{1+n}} (\underline{T}^{N-n})_{s_{1+n} s_1}}{\sum_{s_1} (T^N)_{s_1 s_1}}
\end{aligned}$$

For $h=0$: $$T_{s_i s_{i+1}} = e^{\beta J s_i s_{i+1}}$$
$$= \cosh(\beta J) + s_i s_{i+1} \sinh(\beta J)$$
$$= \cosh(\beta J)\left[1 + s_i s_{i+1} \tanh(\beta J)\right]$$

$$\curvearrowright \left(\underline{T}^2\right)_{s_i s_{i+2}} = \sum_{s_{i+1}} \cosh(\beta J)\left[1+s_i s_{i+1} \tanh(\beta J)\right] \cdot \cosh(\beta J)\left[1+s_{i+1}s_{i+2}\tanh(\beta J)\right]$$

$$= \cosh^2(\beta J)\left\{\left[1+s_i \tanh(\beta J)\right]\left[1+s_{i+2}\tanh(\beta J)\right] + \left[1-s_i \tanh(\beta J)\right]\left[1-s_{i+2}\tanh(\beta J)\right]\right\}$$

$$= 2\cosh^2(\beta J)\left\{1 + s_i s_{i+2} \tanh^2(\beta J)\right\}$$

$$\curvearrowright \left(\underline{T}^\ell\right)_{s_i s_{i+\ell}} = 2^{\ell-1} \cosh^\ell(\beta J)\left[1+s_i s_{i+\ell} \tanh^\ell(\beta J)\right]$$

$$\curvearrowright G(na) = \frac{\sum_{s_1}\sum_{s_{1+n}} s_1 s_{1+n}\, 2^{N-2}\left[1+s_1 s_{1+n} \tanh^n \beta J\right]\left[1+s_{1+n}s_1 \tanh^{N-n}\beta J\right]}{\sum_{s_1} 2^{N-1}\left[1+\underbrace{s_1^2}_{=1} \tanh^N \beta J\right]}$$

$$= \frac{\tanh^n(\beta J) + \tanh^{N-n}(\beta J)}{1+\tanh^N(\beta J)}$$

$$= \tanh^n(\beta J)\,\frac{1+\tanh^{N-2n}(\beta J)}{1+\tanh^N(\beta J)}$$

$\curvearrowright$ For $N \to \infty$ at fixed n:

$$G(na) = \tanh^n(\beta J)$$

(5) As $G(na) \sim e^{-na/\xi}$ (for $T \neq T_c$ and $na \gg \xi$)

we find for the correlation length ξ:

$$-\frac{na}{\xi} = n \ln \tanh(\beta J)$$

$$\xi = -\frac{a}{\ln(\tanh(\beta J))} > 0$$

For high T : $\beta J \to 0$ $\rightsquigarrow \ln(\tanh(\beta J)) \to -\infty$

$\rightsquigarrow \xi \to 0$

For low T : $\beta J \to \infty \rightsquigarrow e^{-\beta J} \to 0$ $\rightsquigarrow \ln(\tanh(\beta J)) = \ln\left(\frac{1-e^{-2\beta J}}{1+e^{-2\beta J}}\right)$

$\approx -2e^{-2\beta J}$

$$\rightsquigarrow \xi = \frac{a}{2} e^{2J/k_B T} \xrightarrow[T\to 0]{} \infty$$

$\rightsquigarrow \xi$ diverges as $T \to T_c$ (critical point)

Solution of Problem 2

(1) We write the partition function in the form

$$Z_N = \sum_{s_1}\sum_{s_2}\dots\sum_{s_N}\left[e^{\tilde{J}(s_1 s_2 + s_2 s_3) + \frac{1}{2}\tilde{h}(s_1 + 2s_2 + s_3)}\right]\left[e^{\tilde{J}(s_3 s_4 + s_4 s_5) + \frac{\tilde{h}}{2}(s_3 + 2s_4 + s_5)}\right]$$

$$\cdot [\dots]\cdot\dots$$

and sum over all even numbered spins

$$= \sum_{s_1}\sum_{s_3}\dots\left[\left(e^{\tilde{J}(s_1+s_3)+\tilde{h}} + e^{-\tilde{J}(s_1+s_3)-\tilde{h}}\right)e^{\frac{1}{2}\tilde{h}(s_1+s_3)}\right]\cdot[\dots]\cdot\dots$$

To have $\tilde{\mathcal{H}}'$ in a similar form as $\tilde{\mathcal{H}}$ (Ising system with next-neighbor interaction) we express the terms $[\dots]$ in the following way

$$\left(e^{\tilde{J}(s_1+s_3)+\tilde{h}} + e^{-\tilde{J}(s_1+s_3)-\tilde{h}}\right)e^{\frac{1}{2}\tilde{h}(s_1+s_3)}$$

$$= \phi(\tilde{J},\tilde{h})\, e^{\tilde{J}' s_1 s_3 + \frac{1}{2}\tilde{h}'(s_1+s_3)}$$

(in (2) we verify that this is possible)

(2) For $s_1 = s_3 = 1$: $\quad \left(e^{2\tilde{J}+\tilde{h}} + e^{-2\tilde{J}-\tilde{h}}\right)e^{\tilde{h}} = \phi\, e^{\tilde{J}'+\tilde{h}'} \qquad (i)$

For $s_1 = s_3 = -1$: $\quad \left(e^{-2\tilde{J}+\tilde{h}} + e^{2\tilde{J}-\tilde{h}}\right)e^{-\tilde{h}} = \phi\, e^{+\tilde{J}'-\tilde{h}'} \qquad (ii)$

For $s_1 = -s_3$: $\quad e^{\tilde{h}} + e^{-\tilde{h}} = \phi\, e^{-\tilde{J}'} \qquad (iii)$

$(i)/(ii) \rightsquigarrow \quad \tilde{h}' = \tilde{h} + \frac{1}{2}\ln\left(\frac{\cosh(2\tilde{J}+\tilde{h})}{\cosh(2\tilde{J}-\tilde{h})}\right) \qquad (iv)$

(iv) and (iii) in $(i) \rightsquigarrow \quad \tilde{J}' = \frac{1}{4}\ln\left\{\frac{\cosh(2\tilde{J}+\tilde{h})\cosh(2\tilde{J}-\tilde{h})}{\cosh^2(\tilde{h})}\right\} \qquad (v)$

$$\phi = \left(16\cosh^2(\tilde{h})\cosh(2\tilde{J}+\tilde{h})\cosh(2\tilde{J}-\tilde{h})\right)^{1/4} \qquad (vi)$$

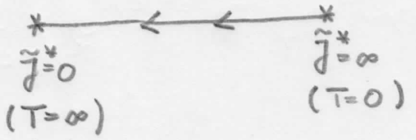
(3) ~~F~~From (v) with $\underline{\tilde{h}=0}$:

$$\tilde{J}' = \frac{1}{2} \ln(\cosh(2\tilde{J}))$$

$$\rightsquigarrow e^{2\tilde{J}'} = \frac{1}{2}(e^{2\tilde{J}} + e^{-2\tilde{J}}) \leq e^{2\tilde{J}}$$

$$\rightsquigarrow \quad \tilde{J}' \leq \tilde{J}$$

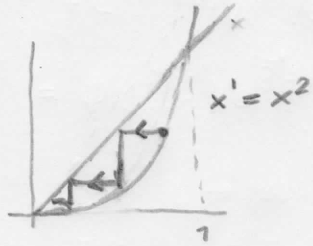
There are 2 fixed points with $\tilde{J}' = \tilde{J}' = \tilde{J}^*$: $\tilde{J}^* = 0, \infty$



The renormalization flow becomes clearer when we rewrite the relation between $\tilde{J}'$ and $\tilde{J}$

$$\tilde{J}' = \frac{1}{2}\ln(\cosh(2\tilde{J})) \iff \tanh\tilde{J}' = \tanh^2\tilde{J}$$

$$x' \quad = \quad x^2$$



(for all starting points $x<1$, the flow goes to zero)

$\rightsquigarrow$ $\tanh\tilde{J}=0 \Rightarrow$ a stable fixed point

~ associated with paramagnetic phase

unstable fixed point at $\tilde{J}=\infty$ describes the Ising critical point at $T=0$

(4) From problem 1 (4,5) we know :

spin-spin correlation function $G(na) = \tanh^n(\tilde{J}) \sim e^{-na/\xi}$

$$\rightsquigarrow \frac{\xi/a}{\xi'/a'} = \frac{\ln(\tanh\tilde{J}')}{\ln(\tanh\tilde{J})} = \frac{\ln(\tanh^2\tilde{J})}{\ln(\tanh\tilde{J})} = 2$$

$$\rightsquigarrow \hat{\xi}' = \frac{1}{2}\hat{\xi} \quad (\text{with } \hat{\xi} \equiv \xi/a)$$

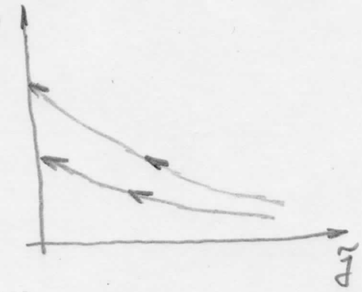
(5) From (iv) follows

$$\frac{\partial \tilde h'}{\partial \tilde h} = 1 + \frac{1}{2} \Big(\underbrace{\tanh(2\tilde J + \tilde h) + \tanh(2\tilde J - \tilde h)}_{>0 \quad (\text{for } \tilde J \neq \infty)} \Big)$$

$$\rightsquigarrow \quad \frac{\partial \tilde h'}{\partial \tilde h} > 1 \quad \text{for all finite } \tilde J$$

$\rightsquigarrow$ field $\tilde h$ becomes larger under iteration

$\rightsquigarrow$ for the renormalization flow



(6) In the vincinity of the critical point $(\tilde J = \infty, \tilde h = 0)$:

From (v) in linear order of $\tilde h$ and $e^{-\tilde J}$

$$\tilde J' = \frac{1}{2} \ln(\cosh 2\tilde J) + \ldots \quad \rightsquigarrow \quad e^{-\tilde J'} = \sqrt{2}\, e^{-\tilde J}$$

From (iv)

$$\tilde h' \approx \tilde h + \frac{1}{2} \ln e^{2\tilde h} = 2\tilde h \qquad \rightsquigarrow \quad \tilde h' = 2\tilde h$$

(flow equations are already decoupled)

(7) From (1) follows: $Z_N(\tilde J, \tilde h) = \left(\phi(\tilde J, \tilde h)\right)^{N/2} Z_{N/2}(\tilde J', \tilde h')$

$$\rightsquigarrow \ln Z_N(\tilde J, \tilde h) = \frac{N}{2} \ln \phi(\tilde J, \tilde h) + \ln Z_{N/2}(\tilde J', \tilde h')$$

$\rightsquigarrow$ free energy per spin $g = (-k_B T) \lim\limits_{N\to\infty} \frac{1}{N} \ln Z_N$

$$\rightsquigarrow \tilde g(\tilde J, \tilde h) \equiv \frac{g(\tilde J, \tilde h')}{k_B T} = -\frac{1}{2} \underbrace{\ln \phi(\tilde J, \tilde h)}_{\text{analytic function}} - \underbrace{\underbrace{\lim_{N\to\infty} \frac{1}{N} Z_{N/2}(\tilde J', \tilde h')}_{\lim\limits_{\frac{N}{2}\to\infty} \frac{1}{2} \frac{1}{N/2} \ln Z_{N/2}(\tilde J', \tilde h')}}_{-\frac{1}{2} \tilde g(\tilde J', \tilde h')}$$

$\rightsquigarrow$ for the singular part of the free energy per spin :

$$g_{sing}(\tilde{J}, \tilde{h}) = \frac{1}{2} \tilde{g}(\tilde{J}', \tilde{h}')$$

$$\rightsquigarrow \quad \tilde{g}_{sing}(e^{-\tilde{J}}, \tilde{h}) = \frac{1}{2} \tilde{g}_{sing}(\sqrt{2}\, e^{-\tilde{J}}, 2\tilde{h})$$

$\rightsquigarrow$ with $\lambda = 2$ as decimation (rescaling) parameter

$$\tilde{g}_{sing}(e^{-\tilde{J}}, \tilde{h}) = \lambda^{-1} \tilde{g}_{sing}(\lambda^{1/2} e^{-\tilde{J}}, \lambda \tilde{h})$$

λ in general arbitrary $\rightsquigarrow$ choose $\lambda^{1/2} e^{-\tilde{J}} = 1$

$$\rightsquigarrow \quad \tilde{g}_{sing}(e^{-\tilde{J}}, \tilde{h}) = e^{-2\tilde{J}} \tilde{g}_{sing}(\tilde{h} e^{+2\tilde{J}})$$

(8) From problem 1 (5) we know

$$(\xi/a)^{-1} = -\ln(\tanh \tilde{J}) = -\ln\left(\frac{1-e^{-2\tilde{J}}}{1+e^{-2\tilde{J}}}\right) \underset{\tilde{J}\to\infty}{\approx} 2e^{-2\tilde{J}}$$

$$\rightsquigarrow \quad \xi = \frac{a}{2} e^{2\tilde{J}} \quad \rightsquigarrow \quad \tilde{g}_{sing}(\xi, \tilde{h}) = \xi^{-1} g_{sing}(\xi \tilde{h})$$

Since $\xi \sim t^{-\nu}$ for $T > T_c$, $h = 0$

For the (singular part of) specific heat at $h=0$

$$c \sim \frac{\partial^2 g_{sing}}{\partial T^2} \sim \xi^{-1+2/\nu} \sim t^{\nu-2} \sim t^{-\alpha} \quad (\text{for } T > T_c)$$

$$\rightsquigarrow \quad -\alpha = \nu - 2 \quad \rightsquigarrow \quad \frac{2-\alpha}{\nu} = 1$$

For the (singular part of) isothermal susceptibility (for $\tilde{h} \to 0$)

$$\chi \sim \frac{\partial^2 g_{sing}}{\partial h^2} \sim \xi^{-1+2} \sim t^{-\nu} \sim t^{-\gamma} \quad (\text{for } T > T_c)$$

$$\rightsquigarrow \quad \nu = \gamma \quad \rightsquigarrow \quad \frac{\gamma}{\nu} = 1$$