Large Fluctuations and Extreme Events, Dresden, October 2015

Power Laws and Criticality

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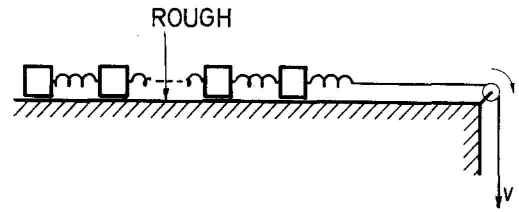
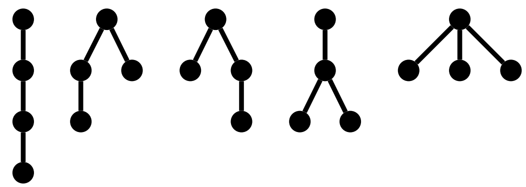
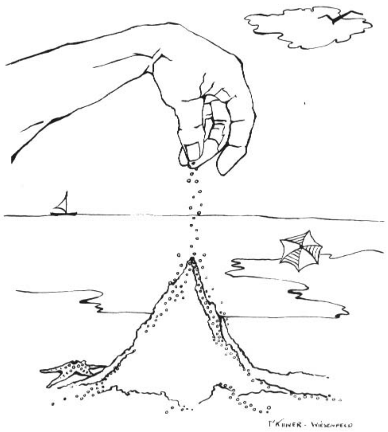
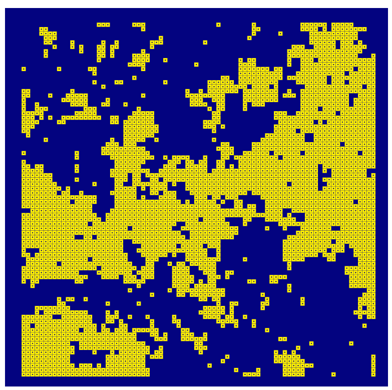
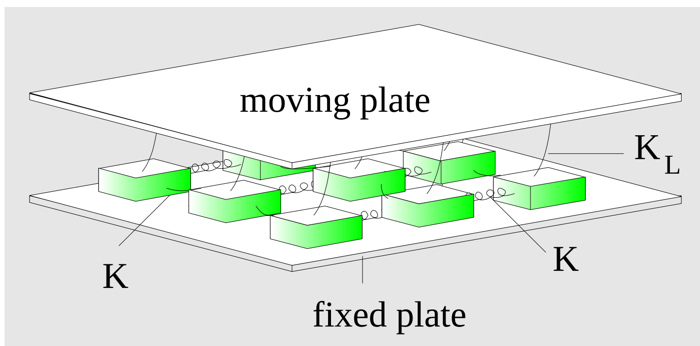










1. **The size of earthquakes** (and other natural hazards)

2. **Properties of power-law distributions**
   - Scale invariance. Divergence of moments

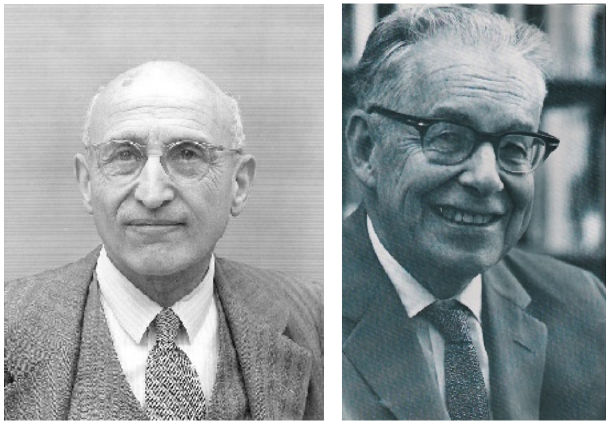
3. **Models for criticality**
   - Galton-Watson model
   - Extinction probability
   - Size distribution

4. **Self-organization towards criticality**
   - Self-organized branching process
   - Manna model, Bak-Tang-Wiesenfeld sandpile model
   - Spring-block earthquake models

5. **Fitting and goodness-of-fit testing of power-law distributions**

Only fools and charlatans predict earthquakes.
C. F. RICHTER

Gutenberg-Richter Law (1941)



seismo.berkeley.edu

- Most important law of statistical seismology and a paradigm of complex-systems geophysics

  For each earthquake with magnitude $m \geq 4$ there are about

  - 0.1 with $m \geq 5$
  - 0.01 with $m \geq 6$, etc...

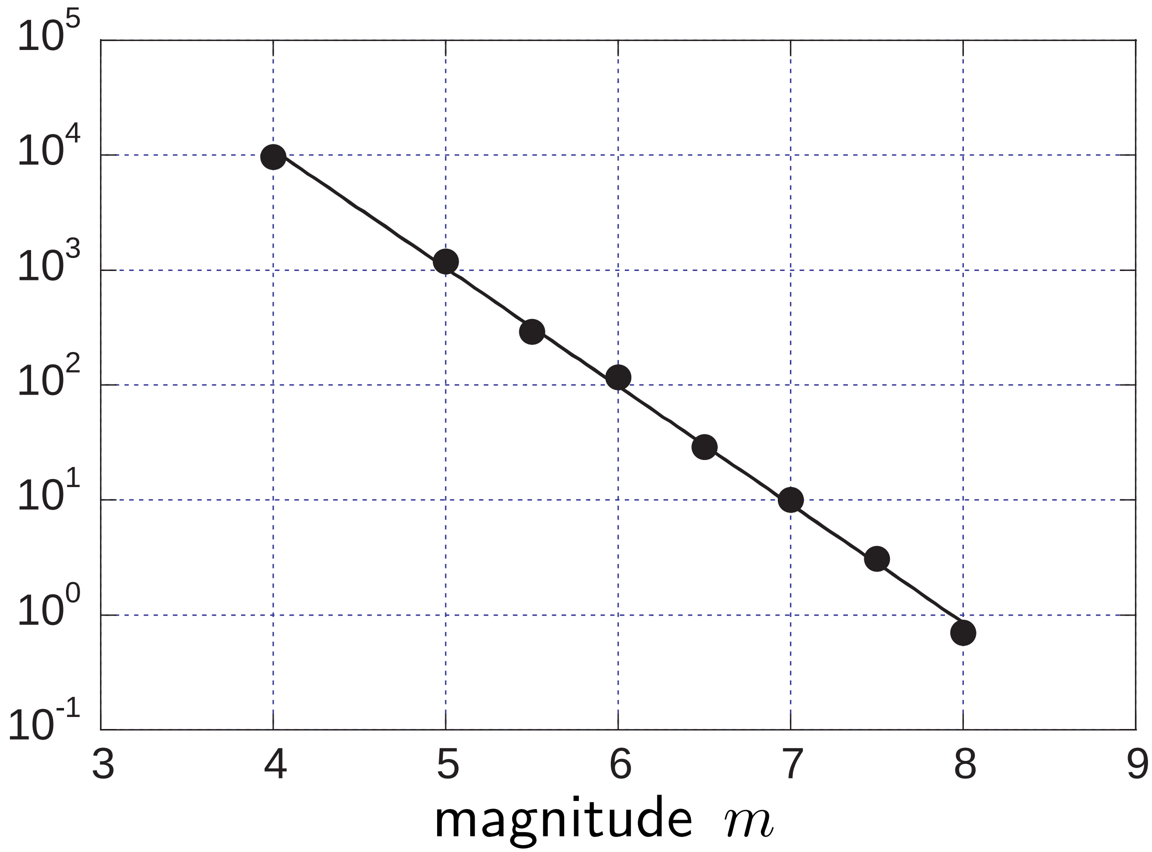
Gutenberg & Richter, BSSA 1944

Number of earthquakes with magnitude $\geq m$

$$N(m) \propto 10^{-bm}, \text{ with } b \simeq 1$$

Good news! $\Rightarrow$ Many small earthquakes, few big ones

- Example: worldwide earthquakes (one-year average)



Kanamori & Brodsky, *Rep Prog Phys* 2004

$$N(m) \propto 10^{-bm} \Rightarrow \log N(m) = \text{constant} - bm$$

Exponential Distribution of Earthquake Magnitudes

- Complementary cumulative distribution (survivor) function

$$S_m(m) = P\{\text{magnitude} \geq m\} \quad \Rightarrow \quad S_m(m) \propto N(m) \propto 10^{-bm}$$

- Probability density

$$D_m(m) = \frac{P\{m \leq \text{magnitude} < m + dm\}}{dm} = -\frac{dS_m(m)}{dm}$$

verifies $\int_0^\infty D_m(m)dm = 1$ and usually has units! $\Rightarrow$ It is not a probability

$$\text{Gutenberg-Richter law} \quad \Rightarrow \quad D_m(m) \propto 10^{-bm}$$

$S_m(m)$ and $D_m(m)$ are "the same" only for the exponential distribution

A statistician would stop here, wouldn't she?

Which is the Meaning of the Gutenberg-Richter Law?

- It depends (of course) on the meaning of magnitude...
  But magnitude is not a proper physical variable (it has no units)!

  Moreover: *magnitudes reflect radiation only from subportions of the rupture, and they saturate above certain size, rather than giving a physical characterization of the entire earthquake source* Ben-Zion, *Rev Geophys* 2008

- Radiated energy is supposed to be an exponential function of magnitude

$$E \propto 10^{3m/2}$$

  (with proportionality factor around 60 kJ)

  An increase of 1 unit in m leads to a factor $\sqrt{10^3} \simeq 32$ in E

$\Rightarrow$ An earthquake with $m = 9$ is “equivalent” to 1000 of $m = 7$

- Then, the Gutenberg-Richter law, in terms of $E \propto 10^{3m/2}$:

$$S_m(m) \propto 10^{-bm} \quad \Rightarrow \quad S_E(E) \propto \frac{1}{E^{2b/3}} \simeq \frac{1}{E^{0.7}}$$

Do you know how to perform the change of variables for the density?

$$D_m(m) \propto 10^{-bm} \quad \Rightarrow \quad D_E(E) \propto \frac{1}{E^{\beta}}$$

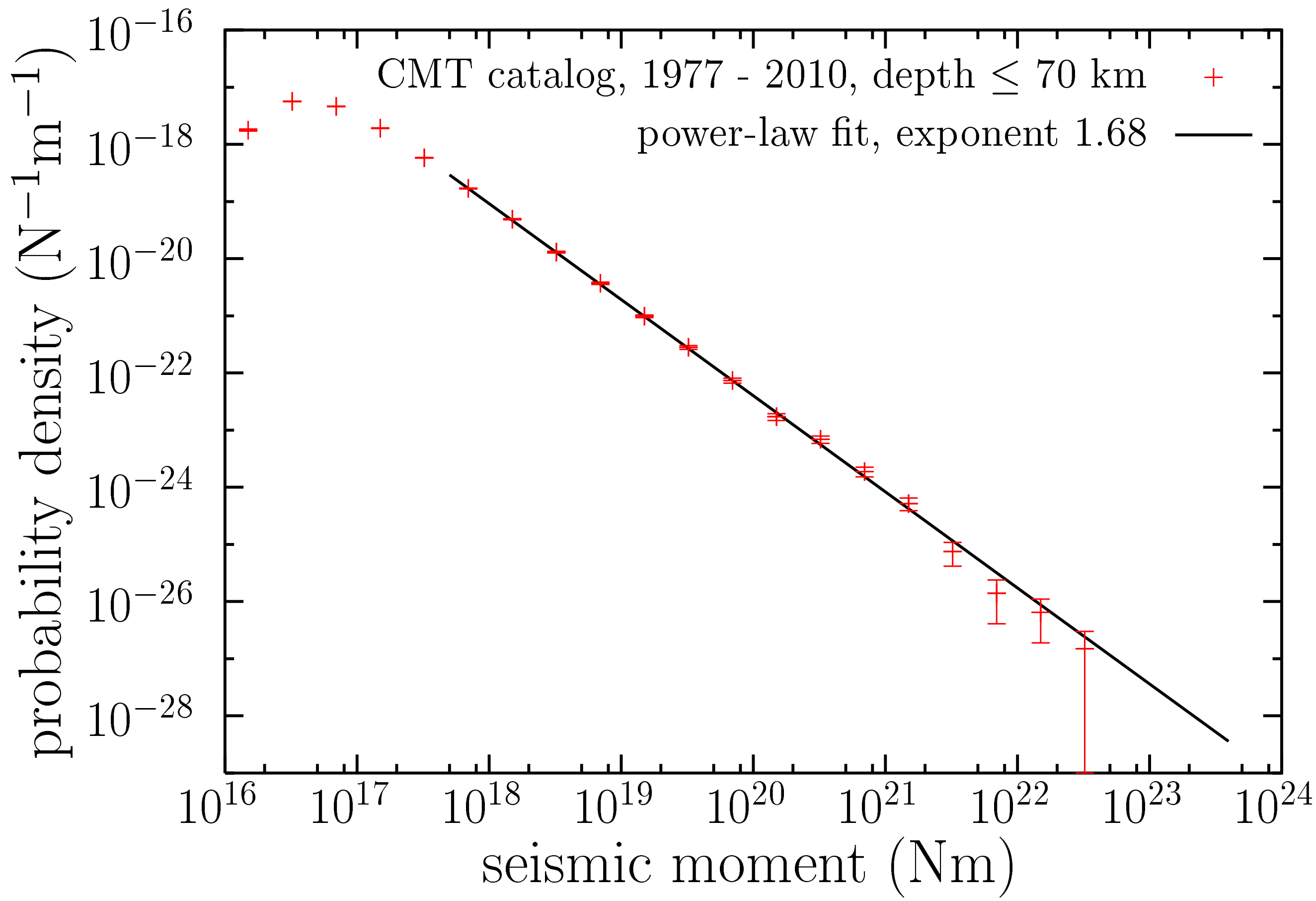
with

$$\beta = 1 + \frac{2b}{3} \simeq 1.7$$

$\Rightarrow$ Earthquake energy is power-law distributed — Wadati 1932, see Utsu, PAGEOPH 1999

$\Rightarrow$ Power-law fit cannot be rejected — Main et al. Nature Geosci. 2008

- Shallow worldwide earthquakes (seismic moment $\sim$ energy):



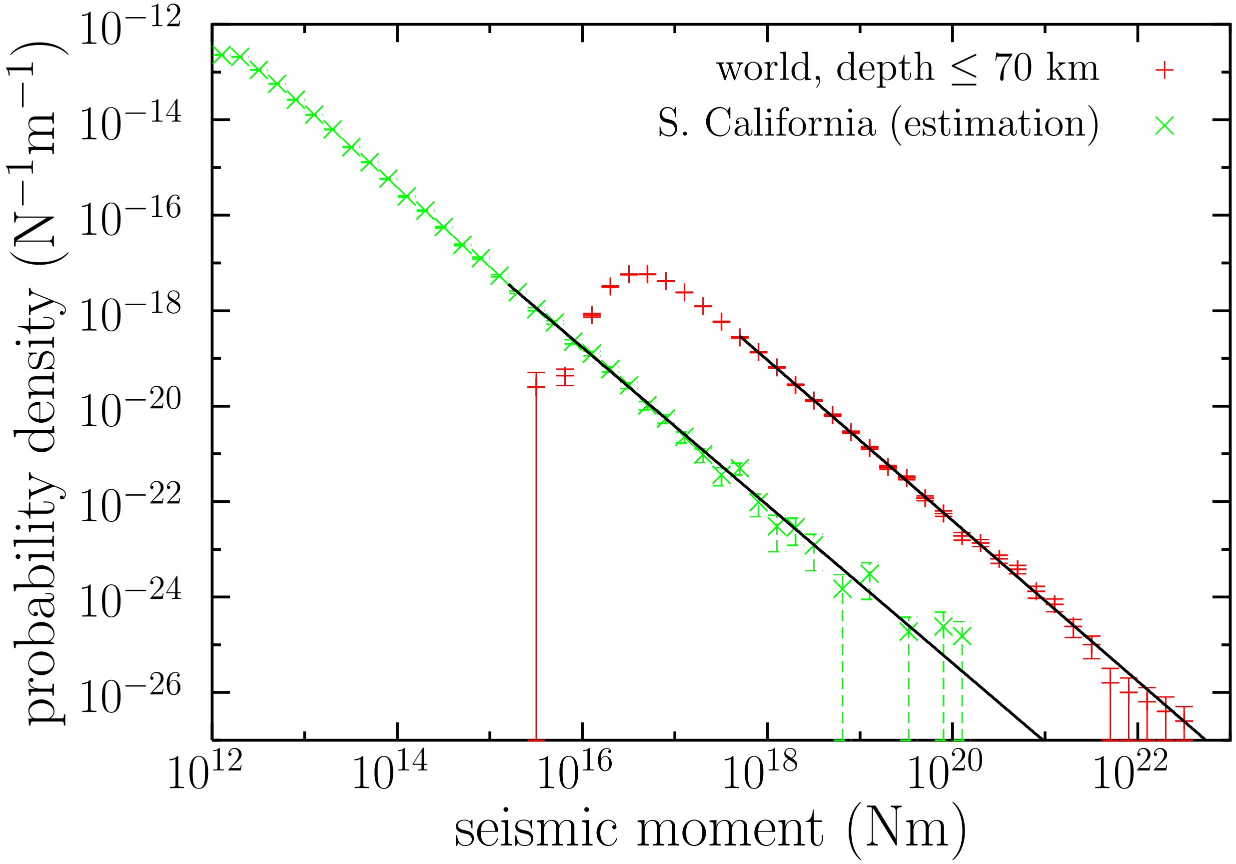
after Kagan

Geophys J Int 2002

Tectonophys 2010

$$D(E) \propto 1/E^{\beta} \quad \Rightarrow \quad \ln D(E) = \text{constant} - \beta \ln E$$

- Compare world with Southern California

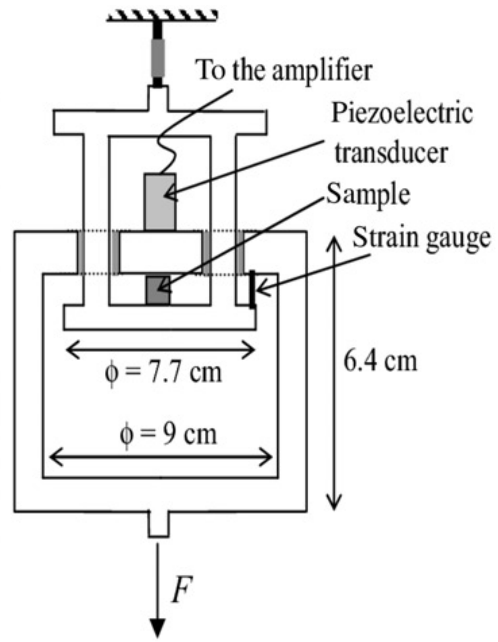


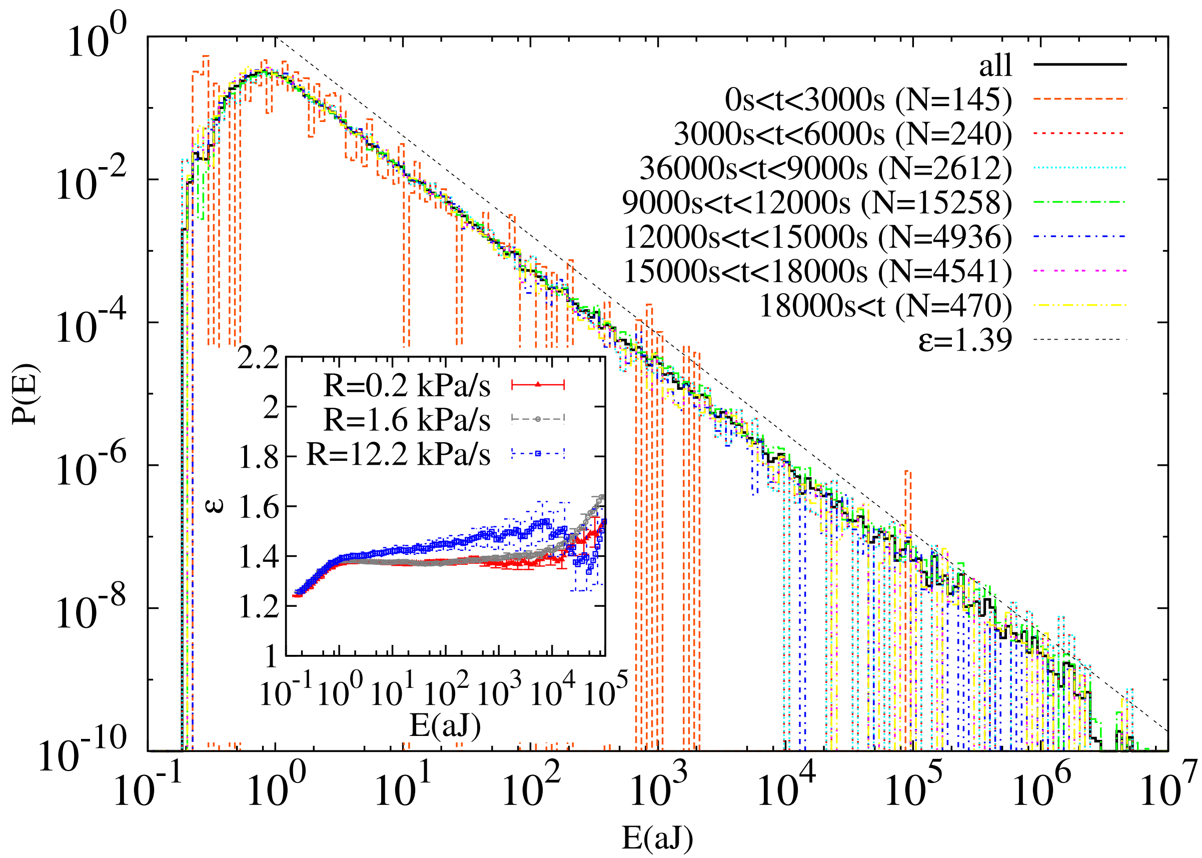
Valid up to $m \simeq -4$ in very small regions

Kwiatek et al., *Bull Seis Soc Am* 2010

- Even valid for fractures in the lab

Baró et al., *Phys Rev Lett* 2013





In nanofractures valid up to $m \simeq -13$

Åström et al. *Phys Lett A* 2006

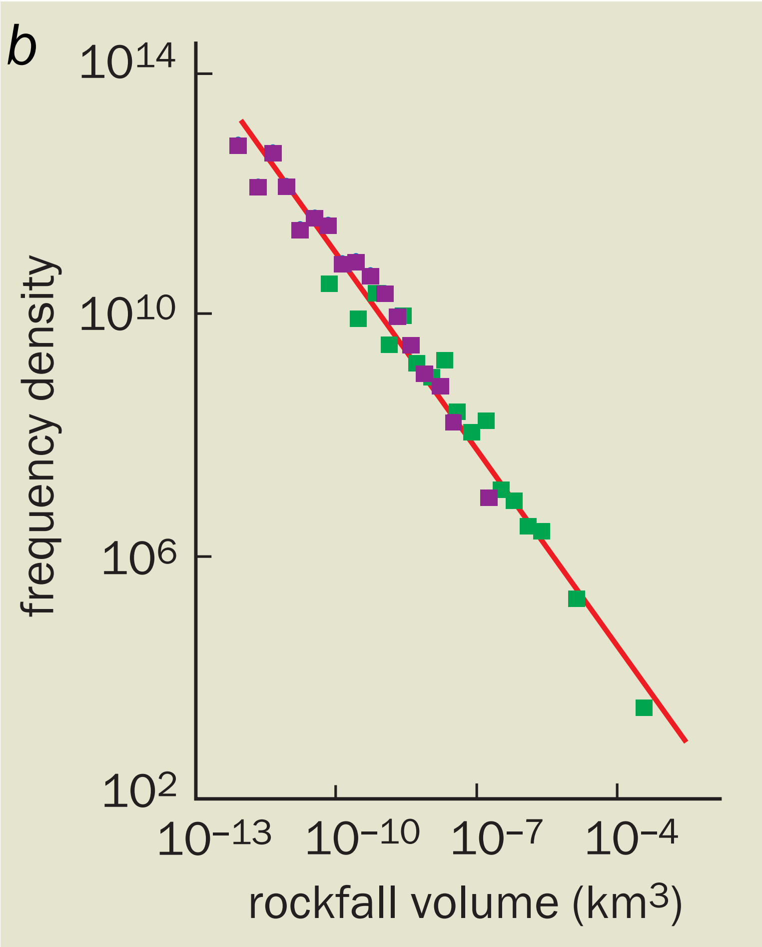
⇒ Enormous range of validity of the Gutenberg-Richter law

- *This law is amazing! How can the dynamics of all the elements of a system as complicated as the crust of the earth, with mountains, valleys, lakes, and geological structures of enormous diversity, conspire, as by magic, to produce a law with such extreme simplicity?*

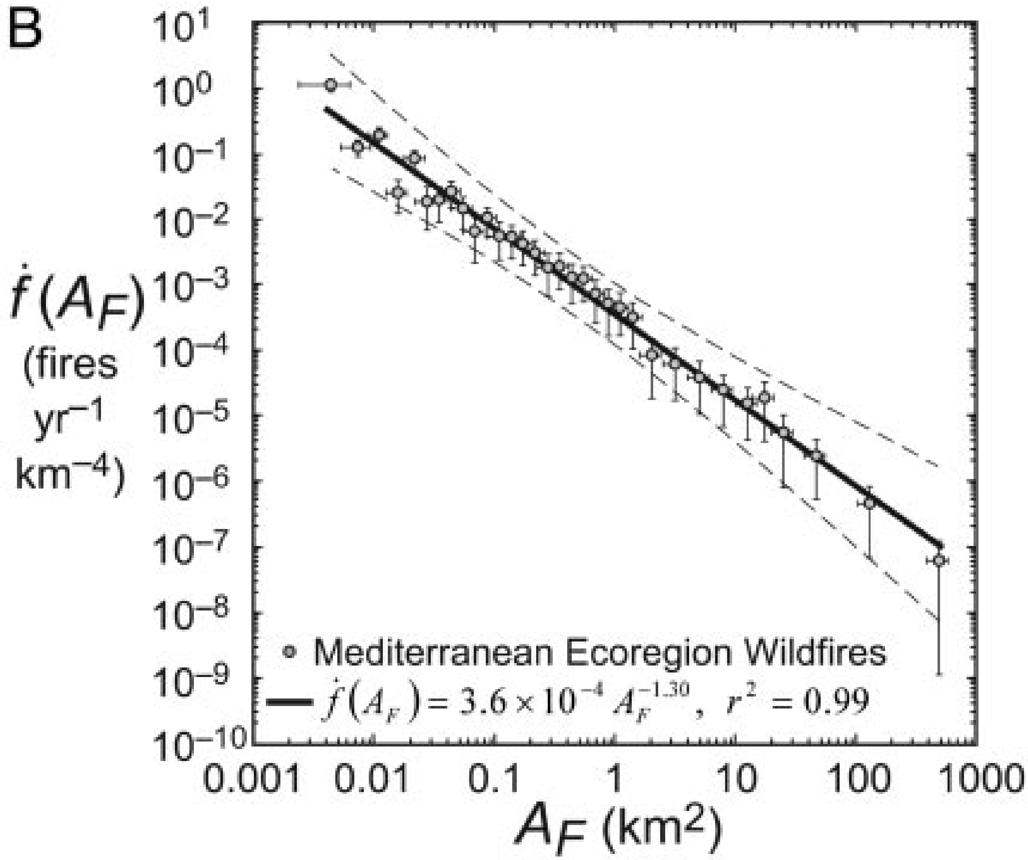
P. Bak, 1996

- Other examples of power-law distributions in natural hazards

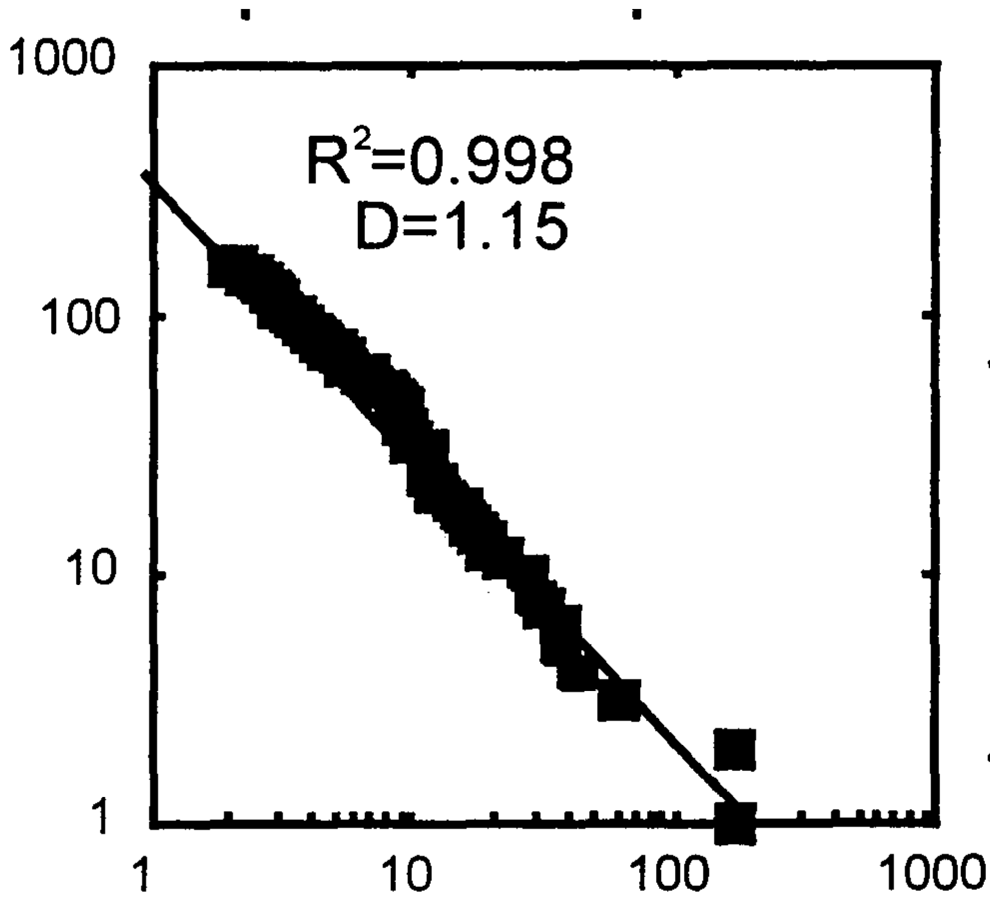
Rockfalls, Malamud, *Phys World* 2004



Forest fires, Malamud et al. *Science* 1998; *PNAS* 2005



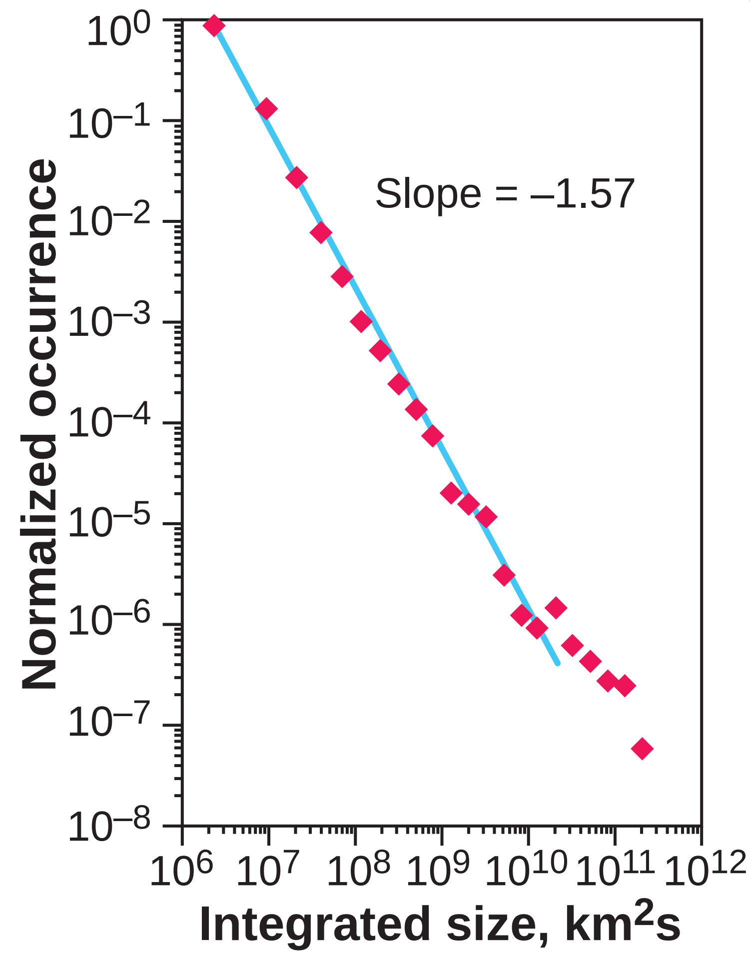
- Volcanic eruptions, Lahaie & Grasso, *J Geophys Res* 1998



Area covered by lava (10^6 m^2)

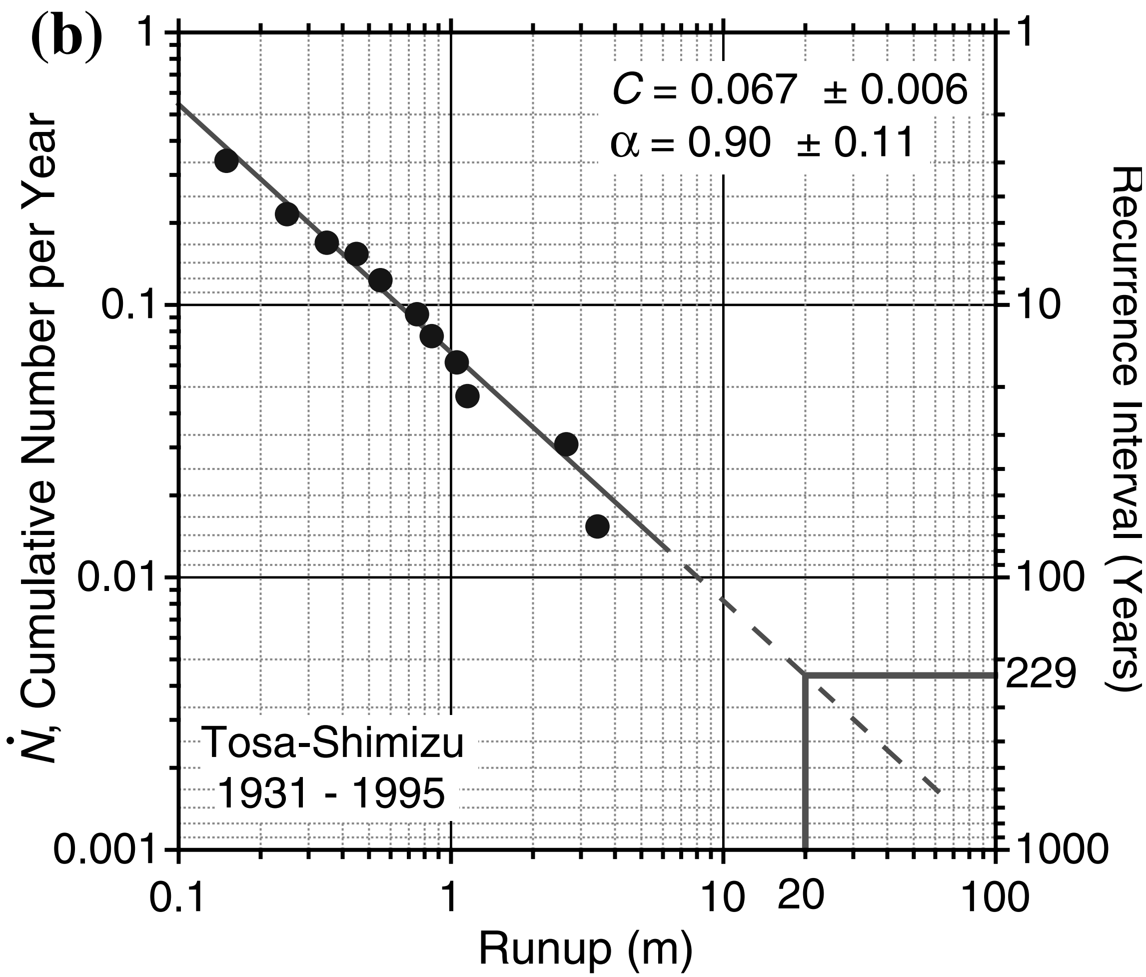
Auroras, Uritsky et al. *J Geophys Res* 2002

Freeman & Watkins, *Science* 2002

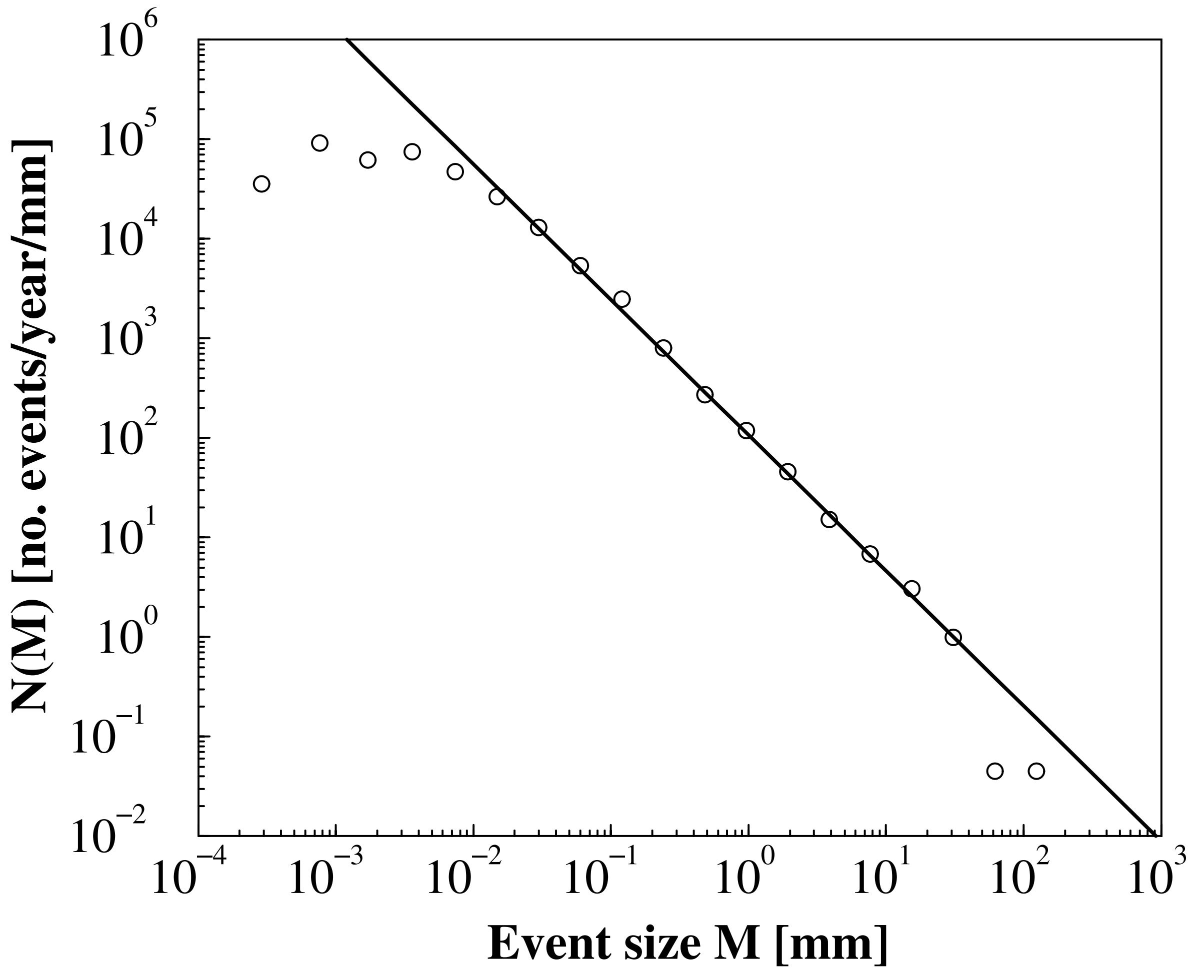


- Tsunamis,

Burroughs & Tebbens, *PAGEOPH* 2005



- Rainfall: flow of water in one point along duration of rain

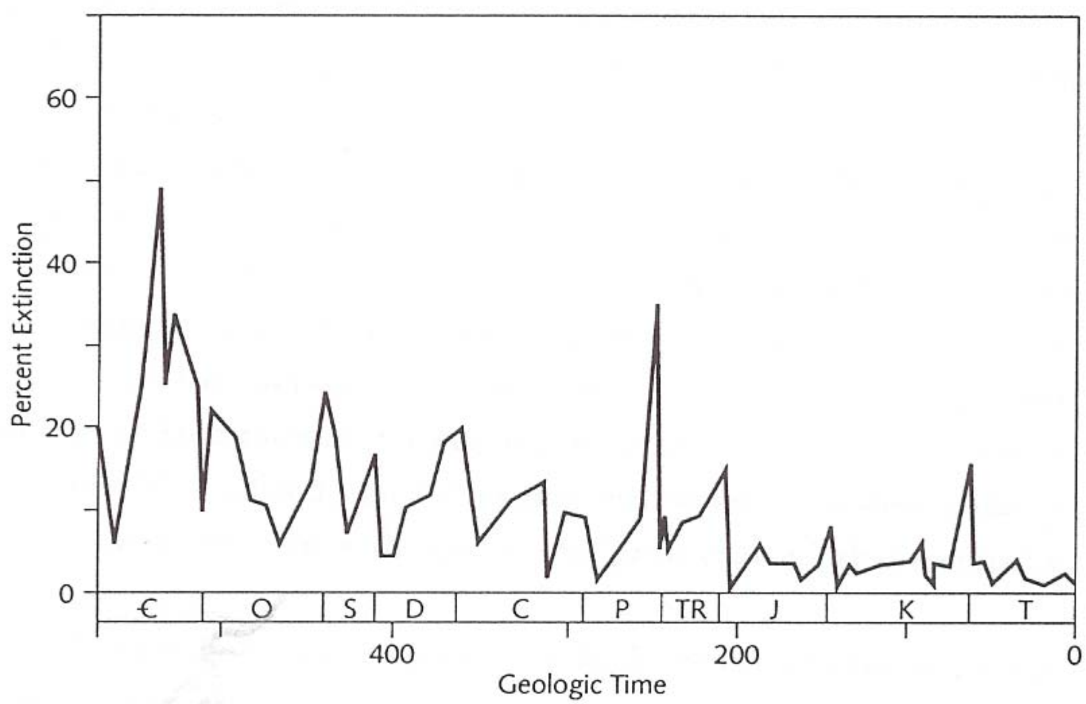


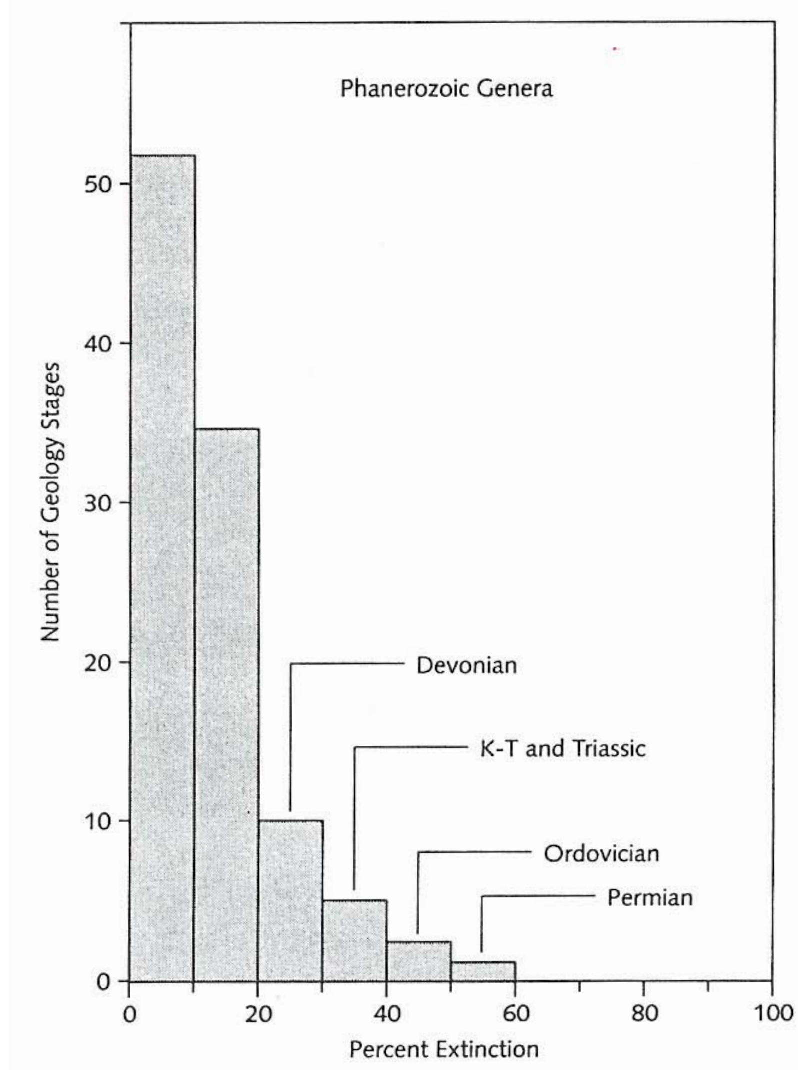
Peters et al. *Phys Rev Lett* 2002; *J Stat Mech* 2010

- Biological extinctions:

  Extinction measured as the percentage of extinct families in fixed periods of time (4 millions years)

  Sepkoski, Raup; after Bak 1996





Scaling laws never happen by accident

G. I. Barenblatt, 2003

Is there anything special about power-law distributions?

Scale transformation

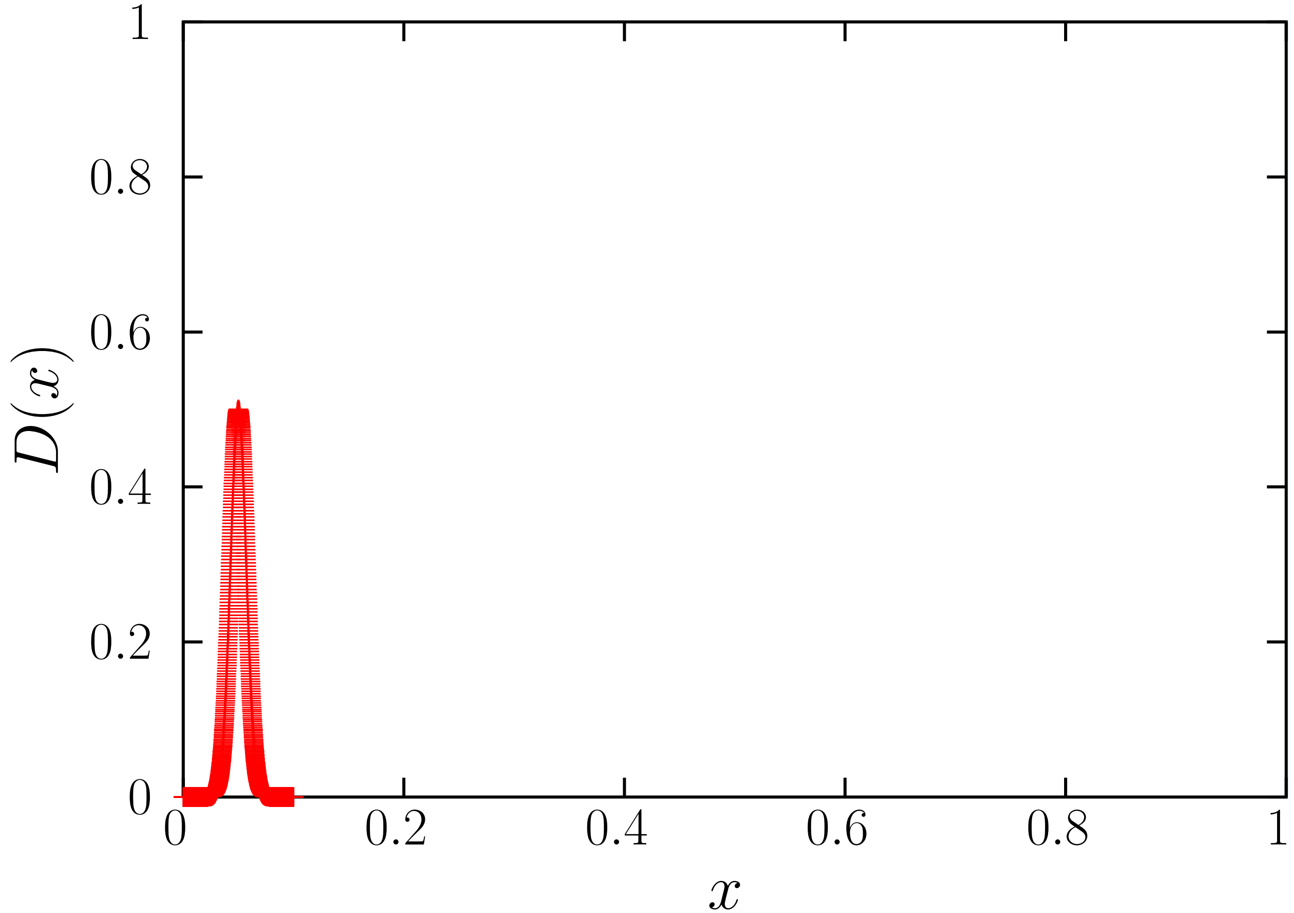
- Consider a function $D(x)$. Let us perform a linear transformation of the axes

$$\top[D(x)] = c_y D(x/c_x)$$

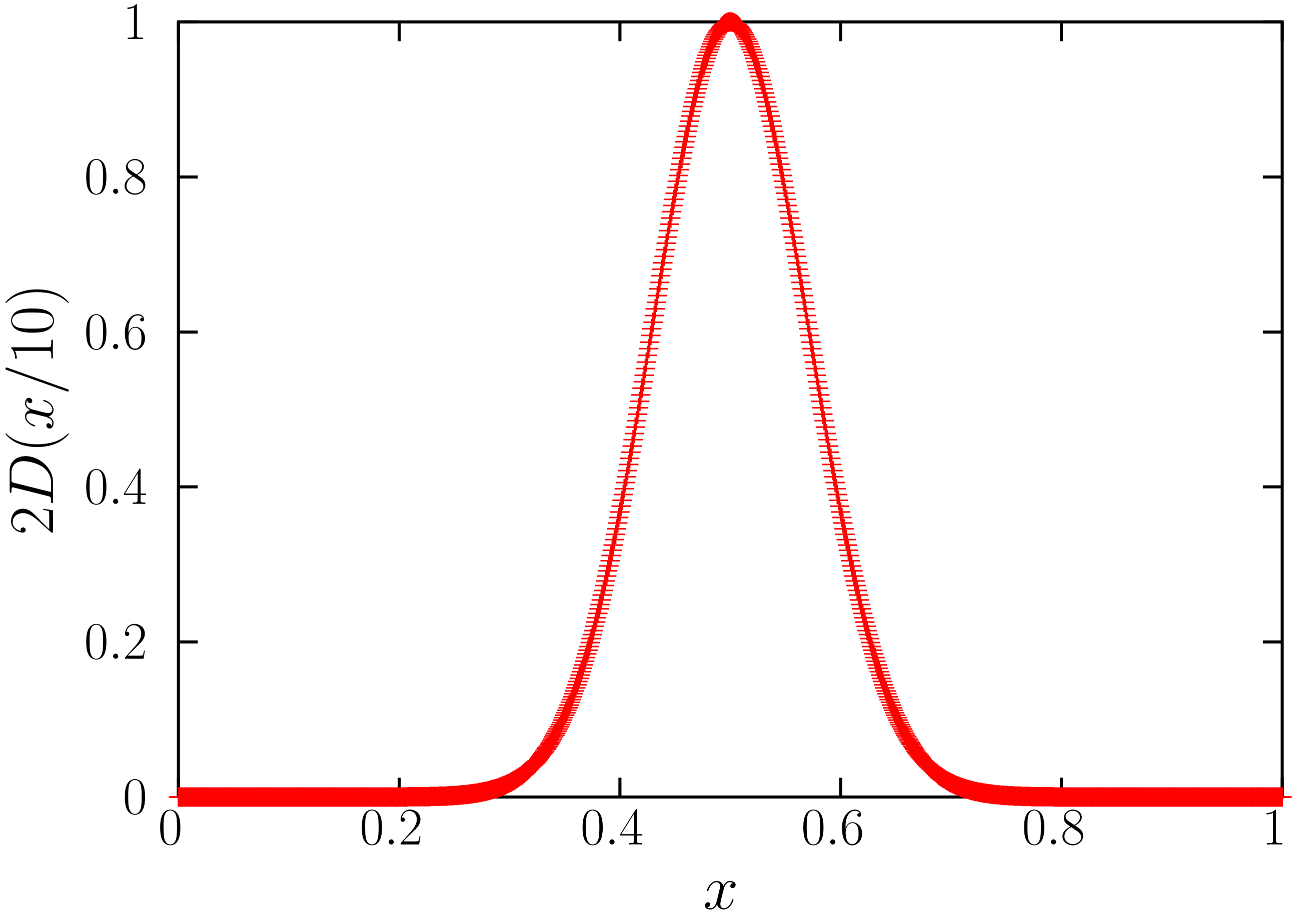
with $c_i > 0$, for $i = x, y$. If $c_i > 1$ then $\top$ acts as a mathematical microscope

- For example:
looking at $D(x)$ at the scale of m, $c_x = c_y = 100$ show $D(x)$ at the scale of cm

- Visual example: $\top[D(x)] = c_y D(x/c_x)$ with $c_x = 10$ and $c_y = 2$



- Visual example: $\top[D(x)] = c_y D(x/c_x)$ with $c_x = 10$ and $c_y = 2$



Scale invariance

- Mathematicians are allowed to ask themselves "silly" questions:

  Invariance under a scale transformation?

$$\top[D(x)] = c_y D(x/c_x) = D(x)$$

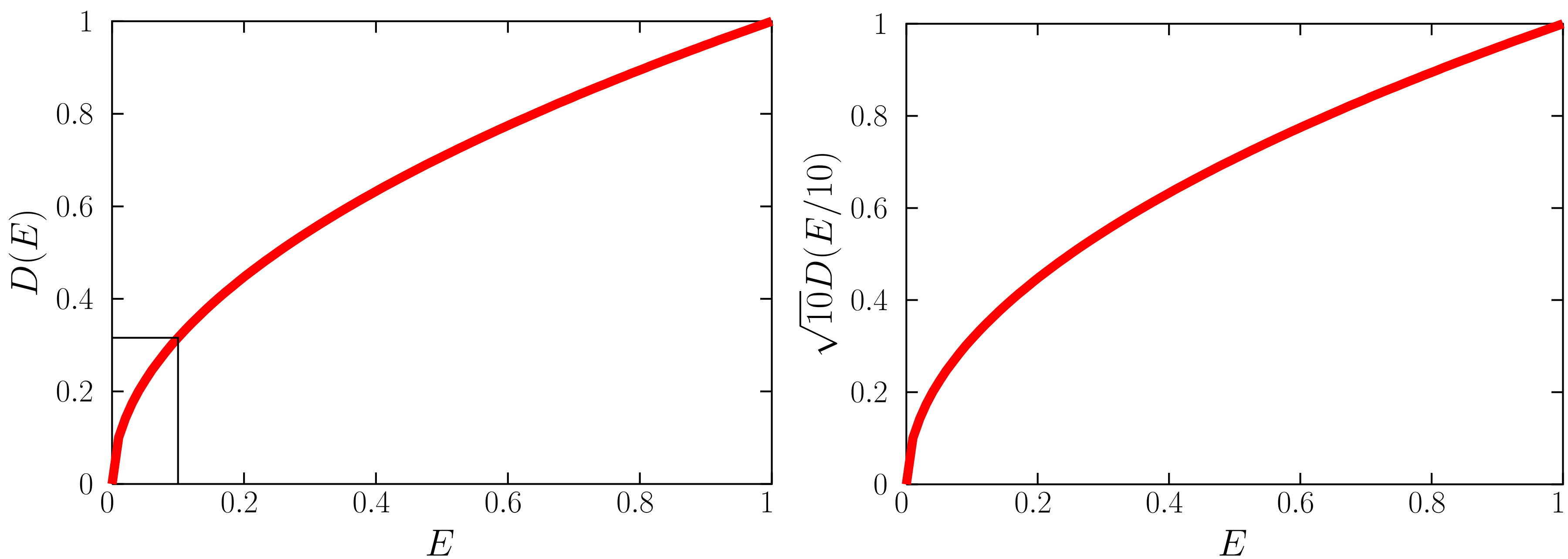
  Solution?

- The only solution of $D(x) = c_y D(x/c_x)$ for all c_x is the power law

$$D(x) \propto \frac{1}{x^\beta} \qquad \text{with } \beta = -\frac{\ln c_y}{\ln c_x} \qquad \text{i.e., } c_y = \frac{1}{c_x^\beta}$$

  Direct substitution confirms that it is a solution indeed

- Example: $D(x) \propto \sqrt{x}$ (i.e., $\beta = -1/2$). If $c_x = 10 \Rightarrow c_y = \sqrt{10}$



Difference between $\beta < 0$ (increasing power law) and $\beta > 0$ (decreasing)

⋆ if $\beta < 0$ and $c_x > 1$ then $c_y = 1/c_x^{\beta} > 1$
⋆ if $\beta > 0$ and $c_x > 1$ then $c_y = 1/c_x^{\beta} < 1$

- Demonstration

Takayasu, *Fractals* 1989; A.C. in *Carpinteri & Lacidogna* 2008

Differentiate both sides of $D(x) = c_y D(x/c_x)$ with respect x and isolate c_y

$$\frac{D'(x)}{D'(x/c_x)/c_x} = c_y = \frac{D(x)}{D(x/c_x)}$$

so, separating variables x and x/c_x and multiplying by x

$$\frac{x D'(x/c_x)}{c_x D(x/c_x)} = \frac{x D'(x)}{D(x)}$$

which has to be valid for all c_x, so, it only can be a constant ($+$, $-$, or 0),

$$\frac{x D'(x)}{D(x)} = \text{constant} = -\beta \qquad \Rightarrow \qquad D(x) \propto \frac{1}{x^\beta} \qquad \text{for } x > 0$$

Meaning of scale invariance?

- Power-law distributions do not have a characteristic scale

  One can define the time unit (or a clock) from the law of radioactive decay (which is an exponential, not a power law)

  But one cannot define a unit of distance from the law of gravitation (which is a power law)

  In the same way that one cannot built a compass from a sphere (which has rotational symmetry)

- So, earthquake energies have no characteristic scale

$\Rightarrow$ It is not possible to answer this simple question:

"How big are earthquakes in this region?"

Implications for extreme events

- We have already seen that the GR law for earthquakes implies that:

  large earthquakes do not play a special role,
  they follow the same law as small earthquakes

  $\Rightarrow$ *general theory encompassing all earthquakes, large and small* P. Bak, 1996

- But scale invariance goes beyond this fact:

  there is no unarbitrary way to separate ordinary events from extreme events

  (at least attending the statistics of event sizes)

Discrete scale invariance

- We can consider the constant β as a complex number, $\beta \to \beta - \omega i$

$$\Rightarrow \frac{1}{x^\beta} \to x^{-\beta+\omega i} = x^{-\beta} e^{i\omega \ln x} \qquad \text{and substitute in } c_y D(x/c_x) = D(x)$$

Then, if c_x is real, then $c_y = 1/c_x^{\beta - i\omega} = c_x^{-\beta} e^{i\omega \ln c_x}$ is complex (in general)

Imposing that c_y is positive real

$$c_x = \exp(2\pi n/\omega) \qquad \text{with } n = 0, \pm 1, \pm 2 \ \ldots$$

Thus, scale invariance does not hold for all c_x but for discrete values

In this case, the real part and the imaginary part are also scale invariant

$$\text{Re}[x^{-\beta+\omega i}] = \frac{1}{x^\beta}\cos(\omega \ln x) \qquad \text{or} \qquad \text{Im}[x^{-\beta+\omega i}] = \frac{1}{x^\beta}\sin(\omega \ln x)$$

Scale invariance for multivariate functions

Christensen & Moloney 2005

- Consider $D(x, y)$ and a scale transformation $\top[D(x,y)] = c_z D(x/c_x, y/c_y)$
  The scale-invariance condition $D(x,y) = c_z D(x/c_x, y/c_y)$ has a unique solution

$$D(x,y) = x^{-\beta} F(y/x^{\alpha}) \qquad \text{for all } c_x > 0$$

  which is called a scaling law, with

$$c_y = c_x^{\alpha} \qquad \text{and} \qquad c_z = \frac{1}{c_x^{\beta}}$$

  and the scaling function? $F()$ is arbitrary

- Equivalent expressions: $D(x,y) = x^{-\beta} F_2(x/y^{1/\alpha}) = y^{-\beta/\alpha} F_3(x/y^{1/\alpha})$, etc.
- We will distinguish scaling laws from power laws
  For univariate functions both are the same, with $F =$ constant

Mean earthquake energy...?

$$E[E] = \langle E \rangle = \int_{min}^{\infty} E D(E) dE \propto \int_{min}^{\infty} \frac{dE}{E^{\beta-1}} = \infty$$

... is infinite! (because $1 < \beta \leq 2$)

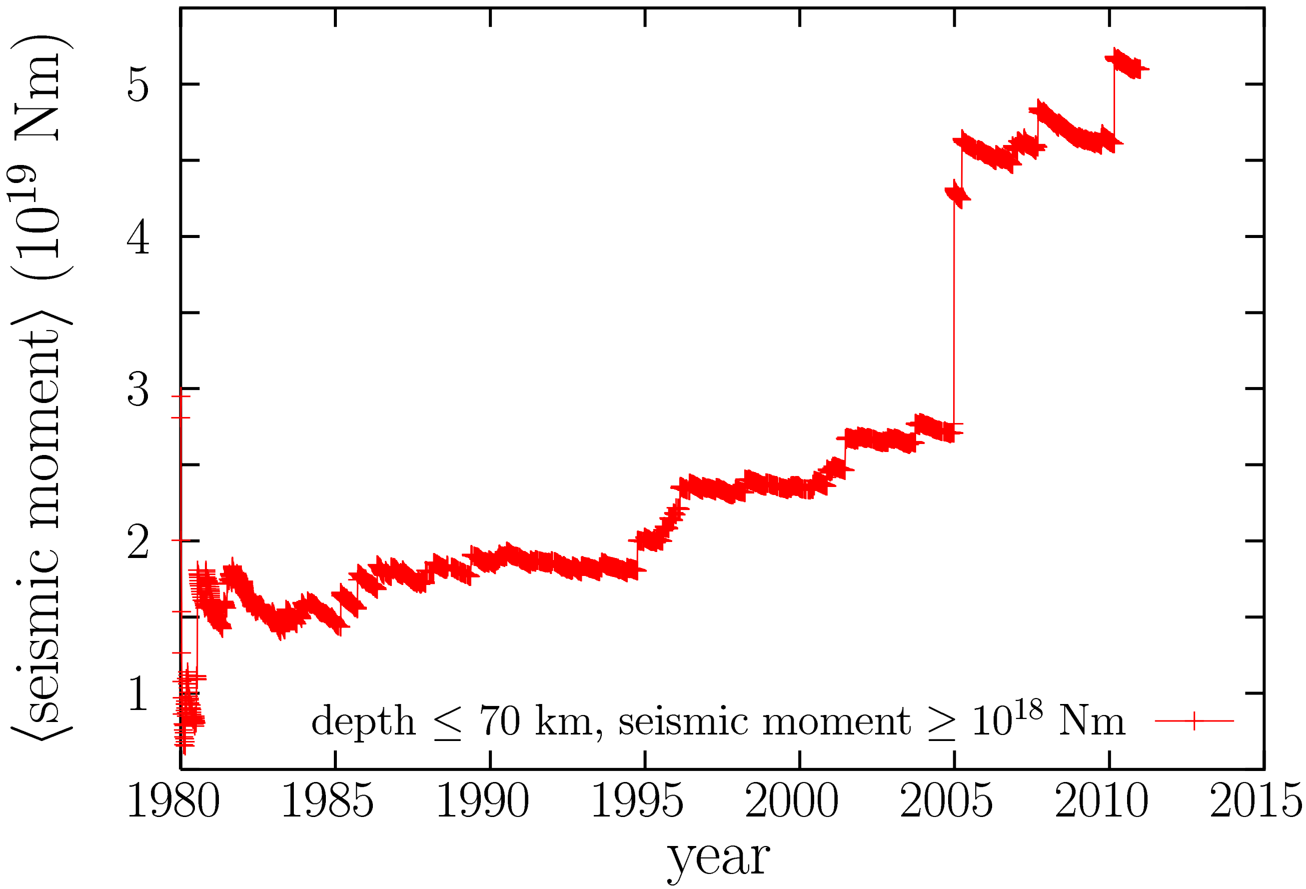
- Higher-order moment are also infinite.
- Which is the problem? Is mathematical?

  This process has a mean waiting time between events which is infinite:

  $$t_{i+1} = t_i + (1-u_i)^{1/(\beta-1)} \text{ with } u_i \text{ uniform random in } [0,1)$$

  Is physical then? The Earth contains a finite amount of energy!
- What does $\langle E \rangle = \infty$ mean in practice?

- Consider the average up to the $N-$th event, $\bar{E} = (E_1 + E_2 + \cdots + E_N)/N$



The rare big events are crucial for energy dissipation $\Rightarrow$ Bad news!!!

Discrete analog: the St. Petersburg paradox

N. Bernoulli 1713 & D. Bernoulli 1738

- Consider a game of chance in which a player tosses a (fair) coin until a tail appears for the 1st time. Each toss doubles the payoff

Outcome	Probability	Payoff
tail	$p_1 = 1/2$	1 \$
heads,tail	$p_2 = 1/4$	2 \$
heads,heads,tail	$p_3 = 1/8$	4 \$
...		
heads ... heads,tail	$p_k = 1/2^k$	2^{k-1} \$ for k tosses, in general

- You are a casino: which would be the fair price to pay to enter the game?

$$\langle \text{payoff} \rangle = \sum_{k=1}^{\infty} p_k \times \text{payoff}(k) = \sum_{k=1}^{\infty} \frac{1}{2^k} \times 2^{k-1}\$ = \frac{1}{2} \times 1\$ + \frac{1}{4} \times 2\$ + \ldots = \infty$$

- Note that the duration k of the game is geometrically (exponentially) distributed

$$p_k = \frac{1}{2^k} = e^{-k \ln 2} = 10^{-k \log 2} \qquad \Rightarrow \qquad \langle K \rangle = \sum_{k=1}^{\infty} p_k k = \frac{1}{1/2} = 2$$

so, the duration of the game is analogous to magnitude, with $b = \log 2 \neq 1$

- But the payoff $= 2^{k-1} \propto 10^{k \log 2} = 10^{ck}$ is analogous to energy, with $c = \log 2$
- Then, the payoff follows a (sort of) discrete power-law distribution with

$$\beta = 1 + \frac{b}{c} = 1 + \frac{\log 2}{\log 2} = 2$$

This is in the "boundary" of having a finite mean

Laplace transform

- Consider $D_x(x)$ defined for $x \geq 0$, then

$$\tilde{D}_x(z) = \int_0^\infty e^{-zx} D_x(x) dx = \langle e^{-zX} \rangle$$

if $D_x(x)$ is a probability density, normalization implies $\tilde{D}_x(z=0) = 1$

- Assuming that $\tilde{D}_x(z)$ exists and that all moments $\langle X^n \rangle$ are finite, and using $e^{-zx} = \sum_{n=0}^{\infty} (-1)^n z^n x^n / n!$

$$\tilde{D}_x(z) = 1 - \langle X \rangle z + \frac{1}{2} \langle X^2 \rangle z^2 - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\langle X^n \rangle z^n}{n!}$$

so, the Laplace transform of $D_x(x)$ is a sort of moment generating function

Sum and rescaling of independent random variables

skip!

- Define $S = X + Y$, then $F_S(s) = P\{\text{sum} < s\} = P\{Y < s - X\} \Rightarrow$

$$F_S(s) = \int_0^s dx \int_0^{s-x} dy D_X(x) D_Y(y) = \int_0^s dx D_X(x) F_Y(s-x)$$

Differentiating with the Leibniz rule, $D_S(s) = dF_S(s)/ds =$

$$= \int_0^s dx D_X(x) \frac{dF_Y(s-x)}{ds} + D_X(x) F_Y(s-x)\Big|_{x=s} = \int_0^s dx D_X(x) D_Y(s-x)$$

Calculating the Laplace transform, with $\theta(x)$ the step function,

$$\tilde{D}_S(z) = \int_0^\infty ds\, e^{-zs} D_S(s) = \int_0^\infty ds\, e^{-zs} \int_{-\infty}^{\infty} dx D_X(x)\theta(x) D_Y(s-x)\theta(s-x)$$

$$\Rightarrow \tilde{D}_S(z) = \int_0^\infty dy\, e^{-zy} D_Y(y) \int_0^\infty dx\, e^{-zx} D_X(x) = \tilde{D}_X(z)\tilde{D}_Y(z) \quad \text{using } s - x = y$$

The sum is a convolution of D_X and D_Y, which turns a product of $\tilde{D}_X$ and $\tilde{D}_Y$

Sum and rescaling of independent random variables

- Define $S = X + Y$, then, the Laplace transform of the distribution of S

$$\tilde{D}_S(z) = \langle e^{-zS} \rangle = \int_0^\infty ds\, e^{-zs} D_S(s) = \int_0^\infty \int_0^\infty dx dy D_X(x) D_Y(y)\, e^{-z(x+y)}$$

where we have used independence $[D_{X,Y}(x,y) = D_X(x) D_Y(y)]$, then

$$\tilde{D}_S(z) = \int_0^\infty dx D_X(x) e^{-zx} \int_0^\infty dy D_Y(y) e^{-zy} = \tilde{D}_X(x) \tilde{D}_Y(y)$$

So, the Laplace transform of the sum is the product of $\tilde{D}_X(z)$ and $\tilde{D}_Y(z)$
It is not necessary to know that $D_S(s)$ is the convolution of $D_X(x)$ and $D_Y(y)$

- In general, if $S = X_1 + X_2 + \cdots + X_N$, then

$$\tilde{D}_s(z) = [\tilde{D}_x(z)]^N$$

when all X_i are independent and identically distributed

- Rescaling by a constant, $R = S/C$

$$\tilde{D}_r(z) = \int_0^\infty dr D_r(r) e^{-zr} = \int_0^\infty ds D_s(s) e^{-z(s/C)} = \tilde{D}_s(z/C)$$

- Defining the rescaled mean, or "non-conserved" average

$$R = \frac{X_1 + X_2 + \cdots + X_N}{N^{1/\alpha}} \qquad \Rightarrow \qquad \tilde{D}_r(z) = [\tilde{D}_x(z/N^{1/\alpha})]^N$$

- Introducing a cumulant generating function

$$G_x(z) = \ln \tilde{D}_x(z) \qquad \Rightarrow \qquad G_r(z) = N G_x(z/N^{1/\alpha})$$

Sum of X's turns into product of m.g.f. and into a sum of cumulant g.f.
(if independence holds)

Note: g.f. $=$ generating function, m.g.f. $=$ moment g.f.

- If the moments are finite (and the generating function exists)

$$\tilde{D}_x(z) = 1 - \langle X \rangle z + \frac{1}{2}\langle X^2 \rangle z^2 - \ldots$$

Considering $\ln(1-y) = -y - y^2/2 - y^3/3 - \ldots$, with $-1 \leq y < 1$, then, the cumulant generating function

$$G_x(z) = \ln \tilde{D}_x(z) = \ln \left[1 - \left(\langle X \rangle z - \frac{1}{2}\langle X^2 \rangle z^2 + \ldots\right)\right] =$$

$$-\left(\langle X \rangle z - \frac{1}{2}\langle X^2 \rangle z^2 + \ldots\right) - \frac{1}{2}\left(\langle X \rangle z - \ldots\right)^2 + \ldots = -\langle X \rangle z + \frac{\langle X^2 \rangle - \langle X \rangle^2}{2} z^2 - \ldots$$

From the coefficients we can obtain the cumulants: $\langle X \rangle$, σ^2, etc.

Distributions stable under "averaging"

- Again a "silly" question: let us look at the fixed points of this transformation

$$G^*(z) = N G^*(z/N^{1/\alpha})$$

This is the scale invariance condition, whose only solution for all N is

$$G^*(z) \propto z^{\alpha}$$

- In the case of the arithmetic mean, $\alpha = 1$, then $D_x^*(x) = \delta(x - \mu)$, indeed

$$\tilde{D}_x^*(z) = \int_0^\infty dx\, e^{-zx} \delta(x - \mu) = e^{-\mu z} \qquad \Rightarrow \qquad G_x^*(z) = \ln D_x^*(z) = -\mu z$$

where $\delta(x - \mu)$ is a Dirac delta "function", which has mean μ and zero variance

Domain of attraction of the Dirac delta distribution

- Considering the expansion of $G_x(z)$ into cumulants (if they exist and are finite)

$$G_x(z) = -\langle X\rangle z + \frac{\sigma^2}{2} z^2 - \ldots$$

Applying the scale transformation we get the distribution of the mean

$$G_{\bar{x}}(z) = G_r(z) = N G_x(z/N) = N\left[-\langle X\rangle \frac{z}{N} + \frac{\sigma^2}{2}\left(\frac{z}{N}\right)^2 - \ldots\right] \to -\langle X\rangle z$$

The distribution of the mean tends to a delta centered at $\langle X\rangle$ when $N \to \infty$
So, the fixed point is attractive if $G_x(z)$ exists and all moments are finite
We will see that the domain of attraction is even bigger

- This constitutes a version of the law of large numbers (weak version) Feller 1971

  It is somehow analogous to the central limit theorem also
  Note that the Gaussian (normal) distribution also tends to a delta
  (because we do not have zero mean)
  If we had subtracted the mean the "central limit" would have been Gaussian

Stability and domain of attraction for "non-conserved" averaging

- Coming back to the general rescaled mean, $G^*(z) \propto z^\alpha$, consider $\alpha = 1/2$

$$G^*_x(z) = -2a\sqrt{z} \qquad \Rightarrow \qquad D^*_x(x) = e^{-a^2/x} \frac{a}{\sqrt{\pi}\, x^{3/2}}$$

Abramowitz & Stegun, 29.3.82; Bouchaud & Georges, *Phys Rep* 1990

As $(x_1 + x_2 + \cdots + x_N)/N^2$ converges, the mean diverges linearly with N

- Do it yourself! Simulate N random values of X. How?
  - Consider the transformation $X = 1/Y^2$
  - Y follows a half-normal (half-Gaussian) distribution
  - Use standard algorithm (like Box-Muller transformation) to simulate Y
- Alternative: simulate a power-law with exponent $3/2$ $\Rightarrow$ What happens?

- Power-law distributions belong to the domain of attraction of $G^*(z) \propto z^\alpha$
  Consider $D_x(x) = B/x^{1+\rho}$ for $x \geq c > 0$ (and 0 otherwise), then $B = \rho c^\rho$ and

$$\tilde{D}_x(z) = B \int_c^\infty e^{-zx} x^{-\rho-1} dx = B z^\rho \Gamma(-\rho, cz)$$

with $\Gamma(\gamma, z) = \int_z^\infty u^{\gamma-1} e^{-u} du$ the incomplete gamma function, with expansion

$$\Gamma(\gamma, z) = \Gamma(\gamma) - z^\gamma \sum_{n=0}^\infty \frac{(-z)^n}{(\gamma+n)n!} \qquad \gamma \neq 0, -1, -2, -3 \ldots \qquad \text{Abramowitz \& Stegun 6.5.29}$$

with $\Gamma(\gamma) = \Gamma(\gamma, 0)$ for $\gamma > 0$ and $\Gamma(\gamma) = \Gamma(\gamma+1)/\gamma$ for $\gamma < 0$ (non-integer)

$$\Rightarrow z^\rho \Gamma(-\rho, z) = z^\rho \Gamma(-\rho) - \sum_{n=0}^\infty \frac{(-z)^n}{(n-\rho)n!} = \frac{1}{\rho}\left[\rho\Gamma(-\rho)z^\rho + \left(1 + \frac{\rho z}{1-\rho} + \ldots\right)\right]$$

$\rho \neq 0, 1, 2, \ldots$ We are interested in $G_x(z) = \ln \tilde{D}_x(z) = \ln B z^\rho \Gamma(-\rho, cz)$, so

$$\ln z^\rho \Gamma(-\rho, z) = -\ln \rho + \ln [\,] = -\ln \rho + \rho\Gamma(-\rho)z^\rho + \frac{\rho z}{1-\rho} + \ldots$$

$$\Rightarrow G_x(z) = \ln \frac{B}{c^\rho} + \ln c^\rho z^\rho \Gamma(-\rho, cz) = \rho\Gamma(-\rho) c^\rho z^\rho + \frac{\rho c z}{1-\rho} + \ldots$$

using again the expansion of the logarithm. Applying the transformation

$$G_r(z) = N G_x(z/N^{1/\alpha}) = N \left[\rho\Gamma(-\rho) c^\rho \left(\frac{z}{N^{1/\alpha}} \right)^\rho + \frac{\rho c}{1-\rho} \left(\frac{z}{N^{1/\alpha}} \right) + \ldots \right]$$

If $0 < \rho < 1$ and $\alpha = \rho$ then $G_r(z) \to \rho\Gamma(-\rho) c^\rho z^\rho$

If $\rho > 1$ and $\alpha = 1$ then $G_r(z) = G_{\bar{x}}(z) \to -\frac{\rho c}{\rho - 1} z$

where the coefficient is the mean of the power-law distribution in this case
(in any other case $G_r(z) \to 0$ or ∞)

- The domain of attraction includes distributions that are asymptotically power laws

$$D_x(x) \sim \frac{B}{x^{1+\rho}} \qquad \text{for } x \to \infty$$

with $B \neq \rho c^{\rho}$. If ρ is not a positive integer

$$\tilde{D}_x(z) \sim B\Gamma(-\rho)z^{\rho} + \sum_{n=0}^{\infty} \frac{a_n(-z)^n}{n!}$$

Bleistein & Handelsman 4.6.23

which is, except for the multiplying constants, the same as before (with $a_0 = 1$)
So, again, there are 2 cases:

If $0 < \rho < 1$	and $\alpha = \rho$	then	$G_r(z) \to B\Gamma(-\rho)z^{\rho}$
If $\rho > 1$	and $\alpha = 1$	then	$G_r(z) = G_{\bar{x}}(z) \to a_1 z$

- Reciprocally, $G_x(z) \propto z^{\rho}$ corresponds to a distribution that is asymptotically power law if $0 < \rho < 1$

Summary

- Assuming independence:

  If the moments of $D_x(x)$ are finite and its generating function exist or

  If $D_x(x)$ is asymptotically a power law with exponent $\beta = 1 + \rho > 2$

  $\Rightarrow$ the arithmetic mean $\bar{x}$ follows a Dirac's delta distribution

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \quad \sim \quad \delta(\bar{x} - \langle x \rangle) \quad \text{when } N \to \infty$$

  $\Rightarrow$ Law of large numbers — Feller 1971

  $\Rightarrow$ Makes sense of the arithmetic mean!

- Assuming independence:
  If $D_x(x)$ is asymptotically a power law with exponent $\beta = 1 + \rho < 2$ (but > 1)

$$\frac{\bar{x}}{N^{1/\rho - 1}} = \frac{1}{N^{1/\rho}} \sum_{i=1}^{N} x_i \quad \text{follows a power-law tailed distribution}$$

  with exponent $1 + \rho$, when $N \to \infty$

⇒ The arithmetic mean diverges as $N^{1/\rho - 1}$ (as $1/\rho - 1 > 0$)

⇒ Case of the generalized central limit theorem   Bouchaud & Georges, *Phys Rep* 1990

- Therefore, the sum of earthquake energies "converges", if rescaled by

$$N^{1/\rho} = N^{1/(\beta - 1)} = N^{3/(2b)}$$

  Standard (conserved) averaging leads to divergence of $\bar{E}$ as $N^{3/(2b) - 1} \simeq \sqrt{N}$

- Concrete example, case $\beta = 1 + \rho = 3/2$

$$D_x(x) \sim \frac{A}{x^{3/2}} \quad \text{for "large" } x$$

The "correct" average to get convergence is

$$\frac{\bar{x}}{N^{1/\rho - 1}} = \frac{1}{N^{1/\rho}} \sum_{i=1}^{N} x_i = \frac{1}{N^2} \sum_{i=1}^{N} x_i = y$$

Then, for $N \to \infty$, the "non-conserved" average y follows

$$D_y(y) = e^{-a^2/y} \frac{a}{\sqrt{\pi}\, y^{3/2}}$$

with $a = -A\Gamma(-1/2)/2 = \sqrt{\pi} A$

What Should One Expect from a Theory of a Complex Phenomenon?

P. Bak, *How Nature Works*, 1996

- A theory must be abstract
  - ⋆ A theory of life does not need to predict elephants (if your theory predicts elephants it is not general enough)
  - ⋆ *Explain why there is variability, or what typical patterns may emerge*
  - ⋆ *If ... we concentrate on an accurate description of the details, we lose perspective*

- A theory must be statistical
  - ⋆ *Collecting anecdotal evidence can only be an intermediate goal.*
  - ⋆ *Anecdotal evidence carries weight only if enough of it can be gathered to form a statistical statement.*
  - ⋆ *Confrontation between theories and ... observations... takes place by comparing the statistical features of general patterns.*

- *The abstractness and the statistical, probabilistic nature of any such theory might appear revolting to geophysicists, biologists, and economists, expecting to aim for photographic characterization of real phenomena.*

- *Perhaps too much emphasis has been put on detailed prediction ...
in today's materialistic world.*

- *To predict the statistics of actual phenomena rather than the specific outcome is a quite legitimate and ordinary way of confronting theory with observations.*

I cannot imagine a theory of earthquakes that does not explain the GR law

Per Bak, 1996

Domino-like theory: Otsuka's model (1971)

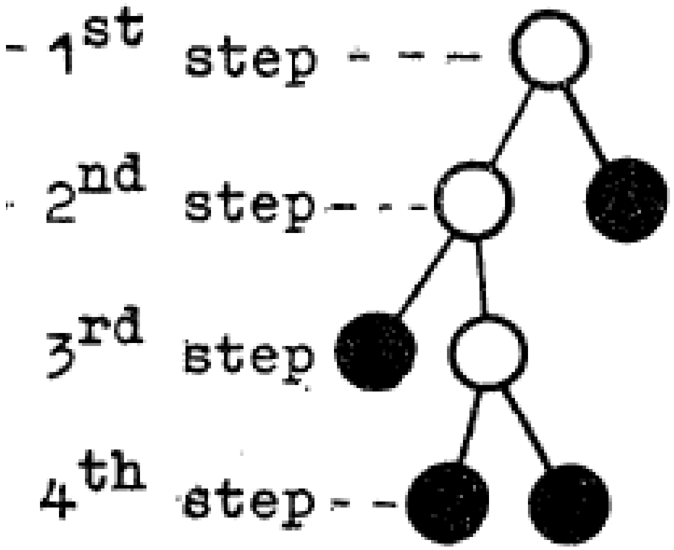
- Earthquake rupture = cascade process of topplings, but:

M. A. Francisco

(1) No domino effect: one toppling does not lead to another one and so on

(2) Pieces are not in a row, rather, in a network or tree, and disordered

⇒ When one piece topples, what happens next is random



appadvice.com

Otsuka, *Zisin* 1971

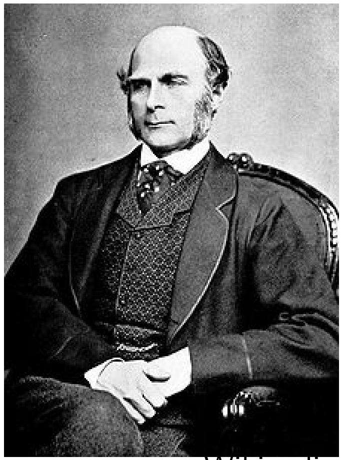
Seismic fault =
patches that may fail and trigger other patches to fail with some probability and so on

Kanamori & Mori, in Boschi et al. 2000

Besides gambling, many probabilists have been interested in reproduction

G. Grimmett and D. Stirzaker, 2001

Galton-Watson (Branching) Process (1873)


Wikipedia


www.wolframalpha.com

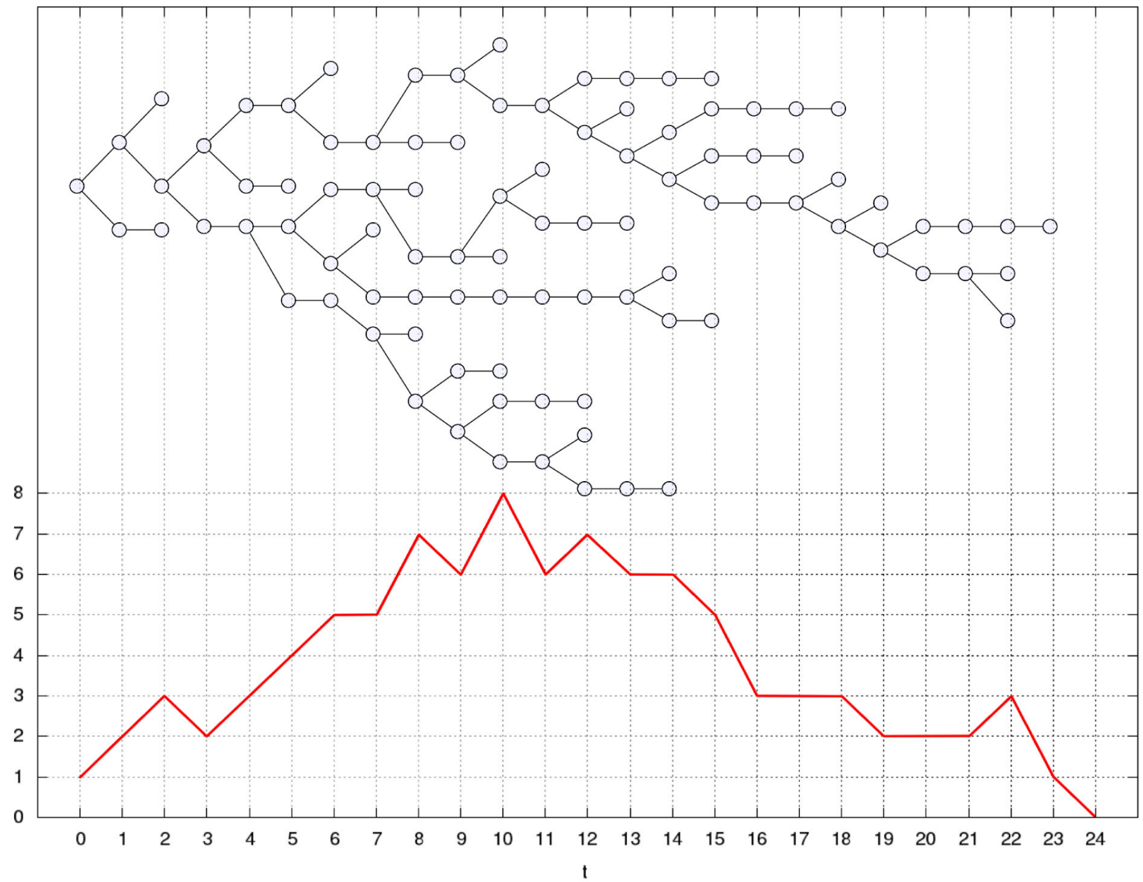
- Definition
  Start with 1 “element” (parent) which generates $K = 0, 1, \ldots$ elements (offsprings) with some probability $p_0, p_1, \ldots$ and so on...

  K's are independent identically distributed

  Galton was not interested in earthquakes
  Rather, he was worried by the extinction of prominent families:
  a rise in physical comfort and intellectual capacity is necessarily accompanied by diminution in “fertility”... If that conclusion be true, our population is chiefly maintained though the “proletariat,” and thus a large element of degradation is inseparably connected with those elements which tend to ameliorate the race

Extinction

- $N_t =$ total number of elements in generation t (with $N_0 = 1$)



Extinction $\Rightarrow$ $N_t = 0$ at some t

- Extinction = extinction in $t = 1$ or in $t = 2$ or $\ldots \in \lim_{t\to\infty}\{N_t = 0\}$

$$\Rightarrow P_{extinction} = \lim_{t\to\infty} P\{N_t = 0\}$$

- Probability generating function of a discrete random variable X

$$f_X(z) = \langle z^X \rangle = \sum_{x=0}^{\infty} P\{X = x\} z^x = P\{X = 0\} + P\{X = 1\} z + \ldots$$

$$\Rightarrow f_X(0) = P\{X = 0\}$$

This is valid for any random variable, also for N_t, so,

$$\Rightarrow P_{extinction} = \lim_{t\to\infty} f_{N_t}(0)$$

which is easier to calculate

- Main equation

$$N_{t+1} = \sum_{i=1}^{N_t} K_i(t)$$

- If N_t were a constant

$$f_{N_{t+1}}(z) = [f_K(z)]^{N_t}$$

Proof:

$$f_{N_{t+1}}(z) = \langle z^{N_{t+1}} \rangle = \langle z^{\sum_i K_i} \rangle = \langle z^{K_1} \dots z^{K_{N_t}} \rangle = \langle z^{K_1} \rangle \dots \langle z^{K_{N_t}} \rangle = [f_K(z)]^{N_t},$$

assuming independence.

- Let us repeat, $N_{t+1} = \sum_{i=1}^{N_t} K_i(t)$. If N_t is constant, $f_{N_{t+1}}(z) = [f_K(z)]^{N_t}$
- But N_t is random, so

$$f_{N_{t+1}}(z) = f_K^{t+1}(z)$$

with $f_K^{t+1}(z) = f_K(f_K(\ldots f_K(z)\ldots)) \; = \;$ composition $t+1$ times

Proof:

$$f_{N_{t+1}}(z) = \langle z^{N_{t+1}} \rangle = \Big\langle \langle z^{N_{t+1}} \rangle_{K_i} \Big\rangle_{N_t} = \langle [f_K(z)]^{N_t} \rangle_{N_t} = f_{N_t}(f_K(z)).$$

As $f_{N_1}(z) = f_K(z)$, then

$$\Rightarrow f_{N_2}(z) = f_{N_1}(f_K(z)) = f_K(f_K(z)) \equiv f_K^2(z)$$

and the result follows by induction

- In conclusion

$$P_{extinction} = \lim_{t \to \infty} P\{N_t = 0\} = \lim_{t \to \infty} f_{N_t}(0) = \lim_{t \to \infty} f_K^t(0)$$

- Let us calculate $f_{N_t}(z)$. Note that $N_1 = K \Rightarrow f_{N_1}(z) = f_K(z)$. Also

$$N_2 = \sum_{i=1}^{N_1} K_i, \qquad \text{and, in general} \qquad N_{t+1} = \sum_{i=1}^{N_t} K_i$$

If $M = \sum_{i=1}^{N} K_i$, with N constant, then

$$f_M(z) = \langle z^M \rangle = \langle z^{\sum_i K_i} \rangle = \langle z^{K_1} \ldots z^{K_N} \rangle = \langle z^{K_1} \rangle \ldots \langle z^{K_N} \rangle = [f_K(z)]^N,$$

assuming independence between the K_i's.
But if N is random, with $f_N(z) = \langle z^N \rangle$, then

$$f_M(z) = \langle z^M \rangle = \Big\langle \langle z^M \rangle_{K_i} \Big\rangle_N = \langle [f_K(z)]^N \rangle_N = f_N(f_K(z)).$$

$$\Rightarrow f_{N_2}(z) = f_{N_1}(f_K(z)) = f_K(f_K(z)) \equiv f_K^2(z)$$

In the same way

$$N_{t+1} = \sum_{i=1}^{N_t} K_i$$

As $f_M(z) = f_N(f_K(z))$, then, $f_{N_3}(z) = f_{N_2}(f_K(z)) = f_K^2(f_K(z)) \equiv f_K^3(z)$
In general, by induction

$$f_{N_t}(z) = f_{N_{t-1}}(f_K(z)) \equiv f_K^t(z) \qquad (t - \text{times composition})$$

Therefore

$$P_{extinction} = \lim_{t \to \infty} P\{N_t = 0\} = \lim_{t \to \infty} f_{N_t}(0) = \lim_{t \to \infty} f_K^t(0)$$

Expected size of population at t

- Property of $f_X(z) = \sum_{x=0}^{\infty} p_x z^x \quad \Rightarrow \quad f'_X(1) = \langle X \rangle$
- Therefore, $\langle N_t \rangle =$

Valid for any generating function, so, in the same way

$$f'_{N_t}(1) = \langle N_t \rangle$$

$$\left.\frac{df_{N_t}(z)}{dz}\right|_{z=1} = \left.\frac{df_K^t(z)}{dz}\right|_{z=1} = \left.\frac{df_K(f_K^{t-1}(z))}{dz}\right|_{z=1} = \left. f'_K(f_K^{t-1}(z))\frac{df_K^{t-1}(z)}{dz}\right|_{z=1}$$

by the chain rule, and by induction

$$\langle N_t \rangle = f'_K(f_K^{t-1}(z)) f'_K(f_K^{t-2}(z)) \cdots f'_K(f_K^2(z)) f'_K(f_K(z))\, f'_K(z)\Big|_{z=1}$$

using $f_K(1) = 1 \Rightarrow f_K^2(1) = 1$, etc., and $f'_K(1) = \langle K \rangle$ then $\langle N_t \rangle = \langle K \rangle^t$

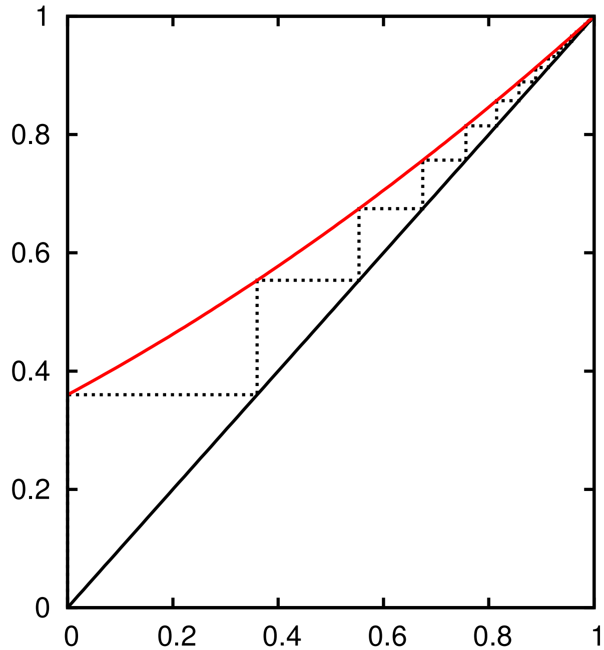
Extinction probability as a function of K

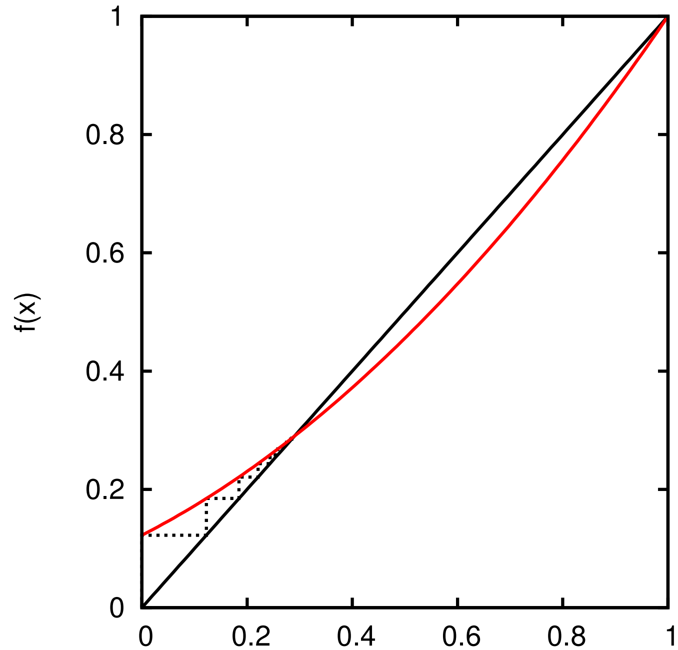
- Properties of $f_K(z)$ in $[0,1]$
  - ⋆ $f_K(0) = p_0$
  - ⋆ $f_K(1) =$
  - ⋆ $f'_K(1) =$
  - ⋆ $f'_K(z)$
  - ⋆ $f''_K(z)$

Extinction probability as a function of K

- Properties of $f_K(z)$
  - $\star$ $f_K(0) = p_0$
  - $\star$ $f_K(1) = 1$
  - $\star$ $f'_K(1) = \langle K \rangle$
  - $\star$ $f'_K(z) \geq 0$
  - $\star$ $f''_K(z) \geq 0$

  Valid for all probability generating functions





$$\langle K \rangle \leq 1 \Rightarrow P_{extinction} = \lim_{t \to \infty} f^t(0) = 1,$$ i.e., extinction for sure

$$\langle K \rangle > 1 \Rightarrow P_{extinction} = \lim_{t \to \infty} f^t(0) = z^* < 1,$$ i.e., non-sure extinction

Except for the "monarchic" case $p_1 = 1$, which has $\langle K \rangle = 1$ but $P_{extinction} = 0$

Phase transition in branching processes

- The fixed point condition for the probability of non-extinction $\rho = 1 - P_{extinction}$,

$$P_{extinction} = 1 - \rho = f_K(P_{extinction}) = f_K(1-\rho) = \sum_{k=0}^{\infty} p_k (1-\rho)^k$$

(because $P\{A\} + P\{no\,A\} = 1$). Expanding using the binomial theorem

$$1 - \rho = \sum_{k=0}^{\infty} p_k \left[1 - k\rho + \frac{1}{2}k(k-1)\rho^2 - \ldots \right] =$$

$$= \sum_{k=0}^{\infty} p_k - \left(\sum_{k=0}^{\infty} p_k k \right) \rho + \frac{1}{2} \left(\sum_{k=0}^{\infty} p_k k(k-1) \right) \rho^2 + \ldots =$$

$$= 1 - \langle K \rangle \rho + \frac{1}{2} \langle K(K-1) \rangle \rho^2 + \ldots$$

- For small ρ (large $P_{extinction}$), introducing $\phi = \langle K(K-1)\rangle$ (2nd factorial moment)

$$\frac{1}{2}\phi\rho^2 - (\langle K\rangle - 1)\rho \simeq 0$$

which has 2 solutions,

$$\rho = 0 \qquad \text{and} \qquad \rho \simeq 2\frac{\langle K\rangle - 1}{\phi}$$

We need to consider the solution closer to (but smaller than) 1, so

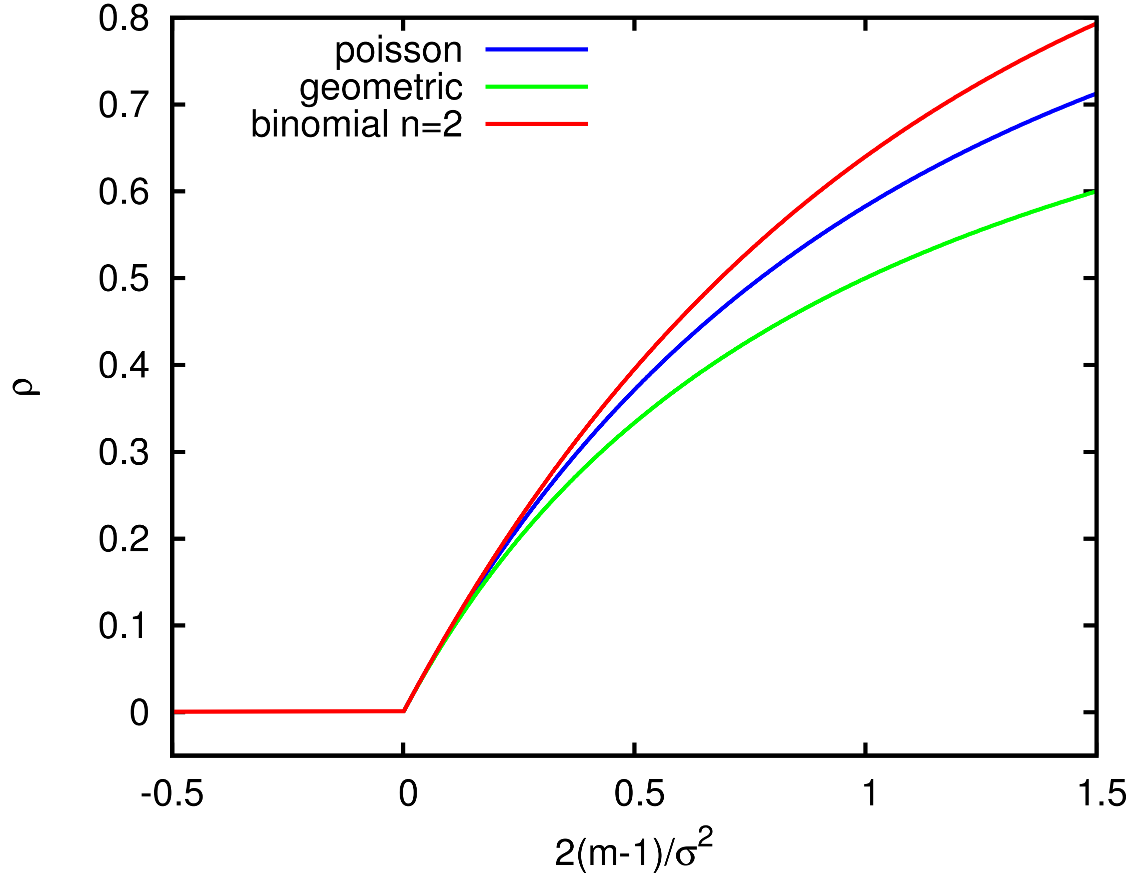
$$\rho = 0 \quad \text{for } \langle K\rangle \leq 1 \qquad \text{and} \qquad \rho \simeq 2\frac{\langle K\rangle - 1}{\sigma_c^2} \quad \text{for } \langle K\rangle \geq 1$$

where we have used $\phi = \sigma^2 + \langle K\rangle(\langle K\rangle - 1)$, if $\rho \simeq 0$ then $\langle K\rangle \simeq 1$ and $\phi \simeq \sigma_c^2$

- The transition is continuous, but sharp $\Rightarrow$ 2nd order phase transition

The case $\langle K\rangle = 1$ is critical, as it separates two very different behaviors

Universality: close to the critical point



$$m = \langle K \rangle$$

Continuous (or second order) phase transition

- Let m be a control parameter ($\langle K \rangle$ in branching or temperature, etc.)

  Let ρ be an order parameter (non-extinction probability, magnetization, etc.)

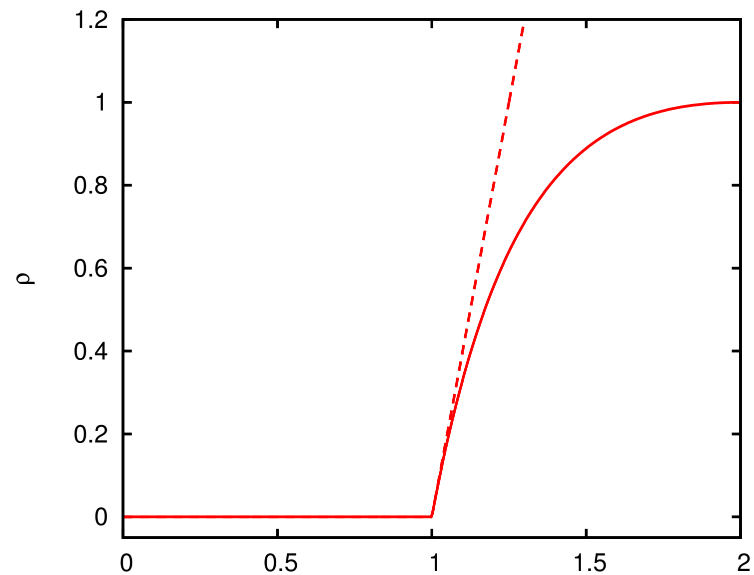
  Then

$$\rho \propto \begin{cases} 0 & \text{for } m \text{ below } m_c = \text{critical point} \\ (m - m_c)^\beta & \text{for } m \text{ above but close to } m_c \end{cases}$$

- Abrupt change in the derivative

  The derivative is discontinuous if $\beta \leq 1$

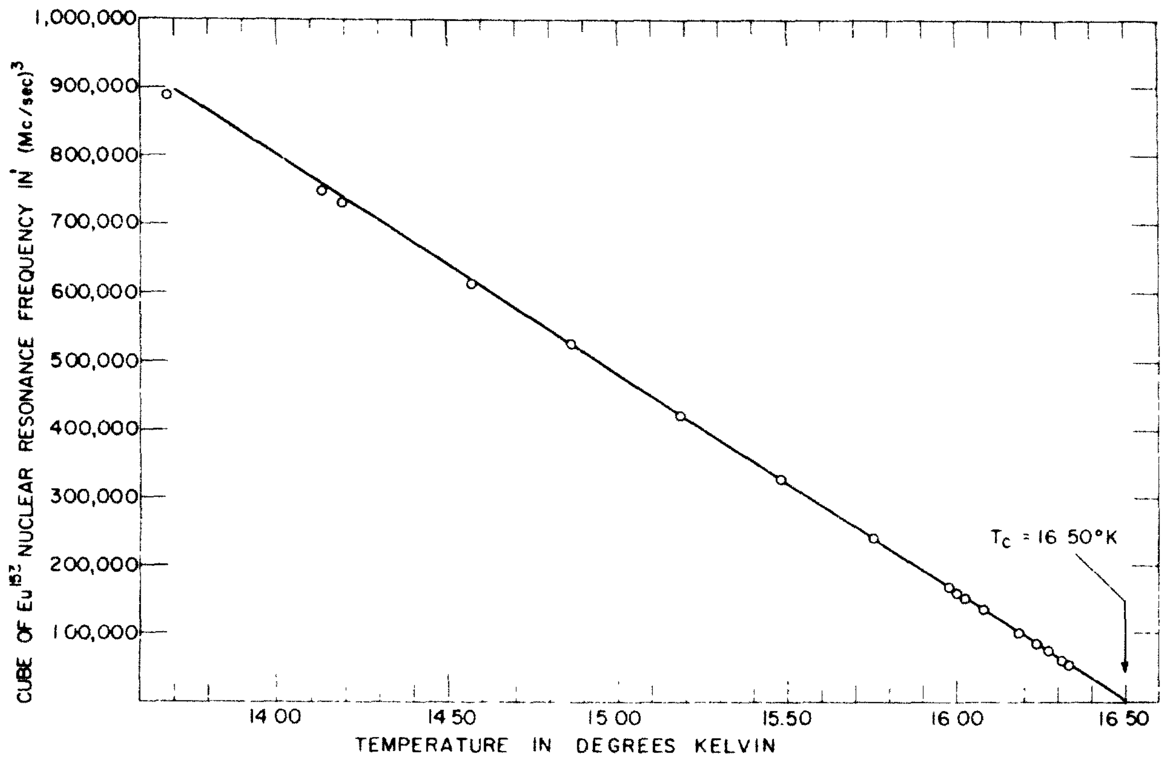
- For a branching process,
  $m_c = 1$ and $\beta = 1$



- For a magnetic system, m is the inverse of the temperature, ρ is magnetization

$\Rightarrow$ m_c is the inverse of Curie temperature and $\beta = 1/3$

Cube of magnet- ization



Magnetization dissappears sharply

Heller & Benedek

Example: binomial number of offsprings

- Each element has only a fixed number of trials n to generate other elements

$$p_k = P\{K = k\} = \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}, \text{ for } k = 0, 1, \ldots n.$$

with p the probability of being successful in each trial, and $q = 1 - p$

- The probability generating function

$$f_K(z) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) q^{n-k} p^k z^k = (q + pz)^n.$$

using the binomial theorem. We will consider $n = 2$

- $P_{extinction}$ will come from the smallest solution in $[0, 1]$ of

$$z^* = (q + pz^*)^2 \Rightarrow z^* = \frac{1 - 2pq \pm \sqrt{(1 - 2pq)^2 - 4p^2q^2}}{2p^2}$$

but for the square root we can write $\sqrt{1 - 4p(1 - p)} = \sqrt{(1 - 2p)^2} = (1 - 2p)$

$$\Rightarrow z = \frac{1 - 2p + 2p^2 \pm (1 - 2p)}{2p^2} = \left\{ \begin{array}{ll} (1 - 2p + p^2)/p^2 & = (q/p)^2 \\ p^2/p^2 & = 1 \end{array} \right.$$

The smallest root depends on whether p is below or above $1/2$

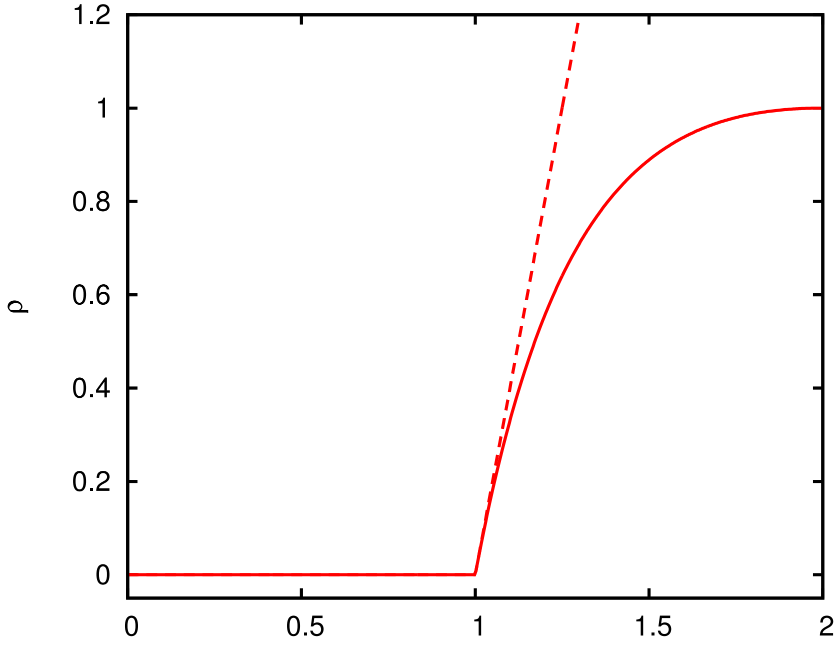
$$P_{extinction} = \begin{cases} 1 & \text{for } p \leq 1/2 \\ (q/p)^2 & \text{for } p \geq 1/2 \end{cases}$$

As $\langle K \rangle = np = 2p$ the critical case $\langle K \rangle = 1$ corresponds to $p = p_c = 1/2$
(in agreement with the behavior of $P_{extinction}$)

- In terms of the non-extinction probability $\rho = 1 - P_{extinction}$

$$\rho = 0 \;\text{ for } p$$

$$\rho = 1 - \left(\frac{q}{p}\right)^2 = \frac{2p-1}{p^2} =$$



using $\langle K \rangle = 2p$

- Expanding around $\langle K \rangle - 1 \simeq 0$

$$\rho \simeq 4(\langle K \rangle - 1) \simeq 2\frac{\langle K \rangle - 1}{\sigma_c^2} \text{ for } p \geq \frac{1}{2}$$

with $\sigma^2 = 2pq$ and $\sigma_c^2 = 1/2$
Then, $p = p_c = 1/2$ or $\langle K \rangle = 1$
is the critical point

Total size of the population

- The size of the population, summing across generations is

$$S = \sum_{t=0}^{\infty} N_t,$$

  - ⋆ total number of individuals that have ever existed, or
  - ⋆ total number of domino pieces toppling,
  - ⋆ "size" of an earthquake, etc...

- Its mean value, for $\langle K \rangle < 1$, using the geometric series, and $\langle N_t \rangle = \langle K \rangle^t$ (new!)

$$\langle S \rangle = \langle N_0 \rangle + \langle N_1 \rangle + \langle N_2 \rangle + \ldots = 1 + \langle K \rangle + \langle K \rangle^2 + \ldots = \frac{1}{1 - \langle K \rangle}$$

Note that when $\langle K \rangle \to 1$, the probability of extinction is 1, but $\langle S \rangle \to \infty$ (!)

Total size of the population: binomial case

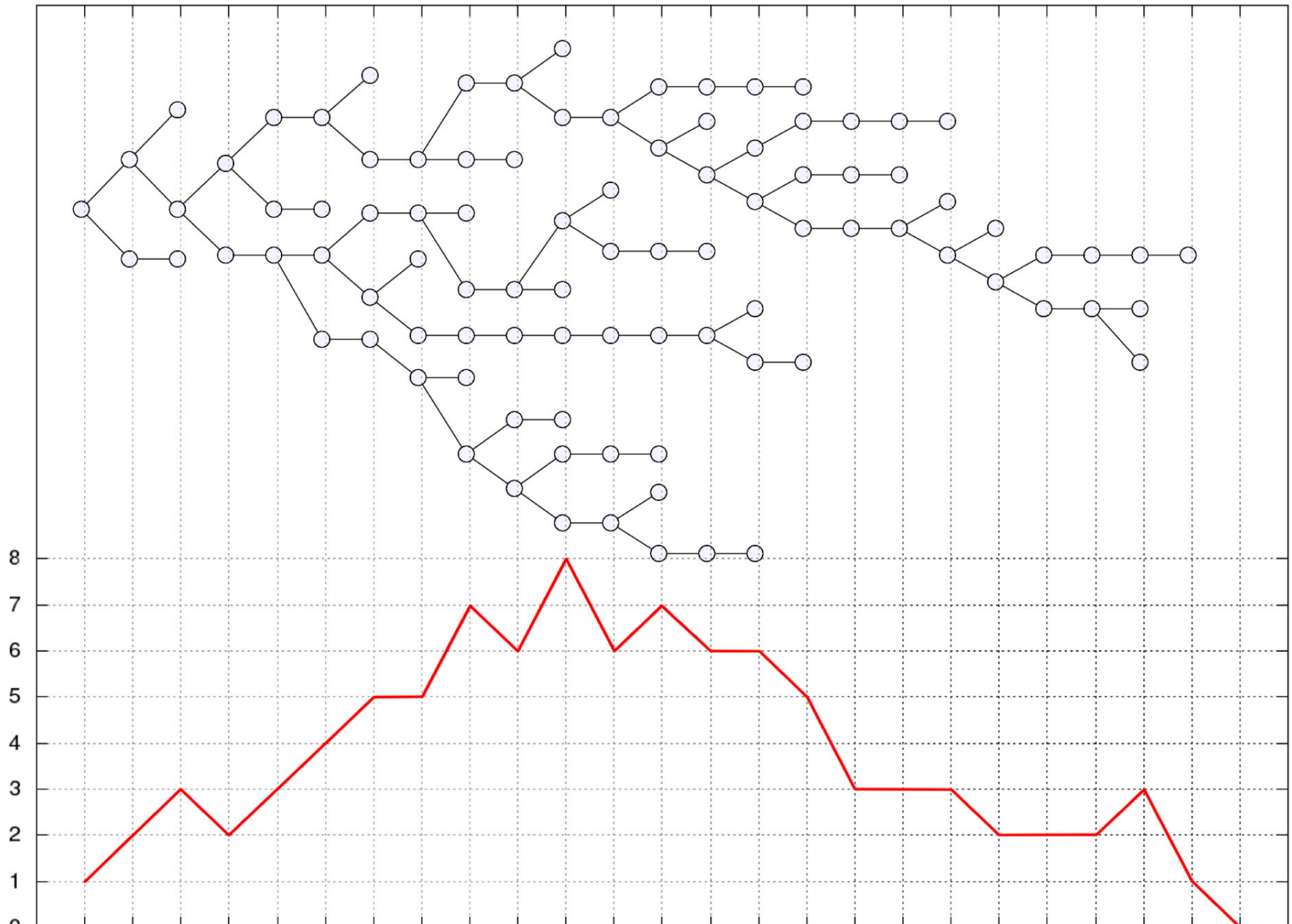
- Each element has only a fixed number of trials n to generate other elements

$$p_k = P\{K = k\} = \binom{n}{k} p^k q^{n-k}, \text{ for } k = 0, 1, \ldots n.$$

with p the probability of being successful in each trial, and $q = 1 - p$

- Remember $\langle K \rangle = np$, so the critical point is at $p_c = 1/n$

- Representation of a branching process as a tree (connected graph with no loops).
  - ⋆ Each element is associated to a node
  - ⋆ Branches linking nodes indicate an offspring relationship between two nodes


8
7
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3
2
1
0

- Representation of a branching process as a tree (connected graph with no loops).
  - ⋆ Each element is associated to a node
  - ⋆ Branches linking nodes indicate an offspring relationship between two nodes

- All nodes have just one incoming branch, except the one in the zero generation
  - ⋆ the number of branches is the number of nodes minus 1, i.e., $s-1$
  - ⋆ the number of possible branches arising from s nodes is ns (in a n-tree)
  - ⋆ the number of missing branches (non-successful trials) is $ns-(s-1)$

  A particular tree of size s comes with a probability

$$p^{s-1}(1-p)^{(n-1)s+1} \text{ with } s=1,2,\ldots$$

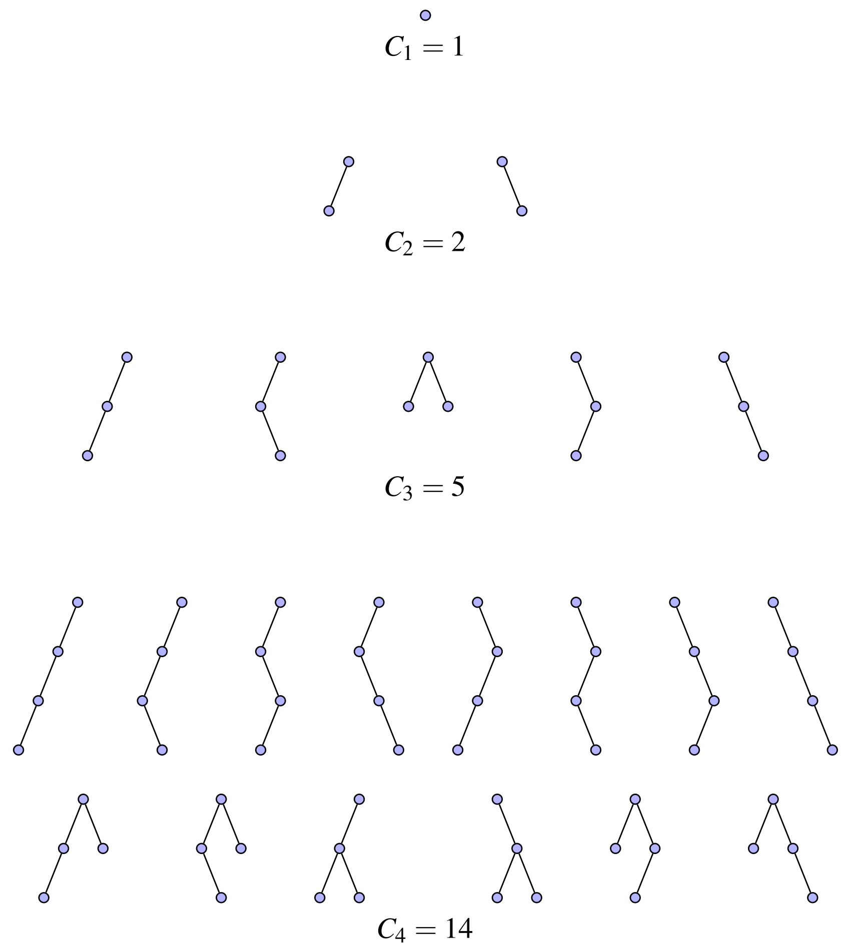
- For $n = 2$, the probability of having an undefined tree of size $s = 1, 2 \ldots$ comes from the Catalan numbers! ...

$$P\{S = s\} = C_s p^{s-1}(1-p)^{s+1} =$$
$$= \frac{1}{s+1}\begin{pmatrix} 2s \\ s \end{pmatrix} p^{s-1}(1-p)^{s+1}$$

with $C_s = \frac{1}{s+1}\begin{pmatrix} 2s \\ s \end{pmatrix}$ the number of different trees of size s, called Catalan numbers

The trees are the internal part of rooted binary trees

Can you draw them?



Calculation of the Catalan numbers

- Let us decompose a tree of size s into its root (zeroth generation) and the rest
  This can be done as
  - ⋆ A subtree of size $s-1$ in the 1st branch and another of size 0 in the 2nd
  - ⋆ A subtree of size $s-2$ in the 1st branch and another of size 1 in the 2nd
  - ⋆ ...
  - ⋆ A subtree of size 0 in the 1st branch and another of size $s-1$ in the 2nd

  So, the total number of trees of size s is

$$C_s = C_0 C_{s-1} + C_1 C_{s-2} + \cdots + C_{s-2} C_1 + C_{s-1} C_0 \qquad \text{with } C_0 = 1$$

- We define a generating function for the Catalan numbers

$$h(x) = C_0 + C_1 x + C_2 x^2 + \ldots = \sum_{s=0}^{\infty} C_s x^s$$

The properties of the Catalan numbers will allow the calculation of $h(x)$

$$[h(x)]^2 = \sum_{i,j=0}^{\infty} C_i C_j x^{i+j} = \sum_{s=0}^{\infty} \underbrace{\left[\sum_{i+j=s} C_i C_j\right]}_{C_{s+1}} x^s = \frac{1}{x}\sum_{s=0}^{\infty} C_{s+1} x^{s+1} = \frac{h(x) - C_0}{x}$$

so

$$h(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

but this tell us nothing yet. Using the Taylor expansion of $\sqrt{1-x}$

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{1}{4}\left(\frac{x^2}{2!}\right) - \frac{3}{8}\left(\frac{x^3}{3!}\right) - \ldots = 1 - \frac{x}{2} - \sum_{s=1}^{\infty} \frac{(2s-1)!!}{2^{s+1}(s+1)!} x^{s+1}$$

then

$$\sqrt{1-4x} = 1 - 2x - \sum_{s=1}^{\infty} \frac{(2s-1)!!2^{s+1}}{(s+1)!} x^{s+1}$$

and so, taking the minus sign (otherwise $h(x)$ is not a g.f.)

$$h(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + \frac{1}{2x}\sum_{s=1}^{\infty} \frac{(2s-1)!!2^{s+1}}{(s+1)!} x^{s+1} = 1 + \sum_{s=1}^{\infty} \frac{(2s-1)!!2^{s}}{(s+1)!} x^{s}$$

then the Catalan numbers are, and using $(2s)! = (2s)!!(2s-1)!! = s!2^s(2s-1)!!$

$$C_s = \frac{(2s-1)!!2^s}{(s+1)!} = \frac{(2s)!}{s!(s+1)!} = \frac{1}{s+1}\begin{pmatrix} 2s \\ s \end{pmatrix}$$

the latter being valid for $s = 0, 1, 2 \ldots$

Coming back to the Taylor expansion of $\sqrt{1-4x}$

$$\sqrt{1-4x} = 1 - 2x \sum_{s=0}^{\infty} C_s x^s$$

Parenthesis: many uses of the Catalan numbers

Davis, *geometer* 2010

- Number of balanced configurations with n pairs of parenthesis

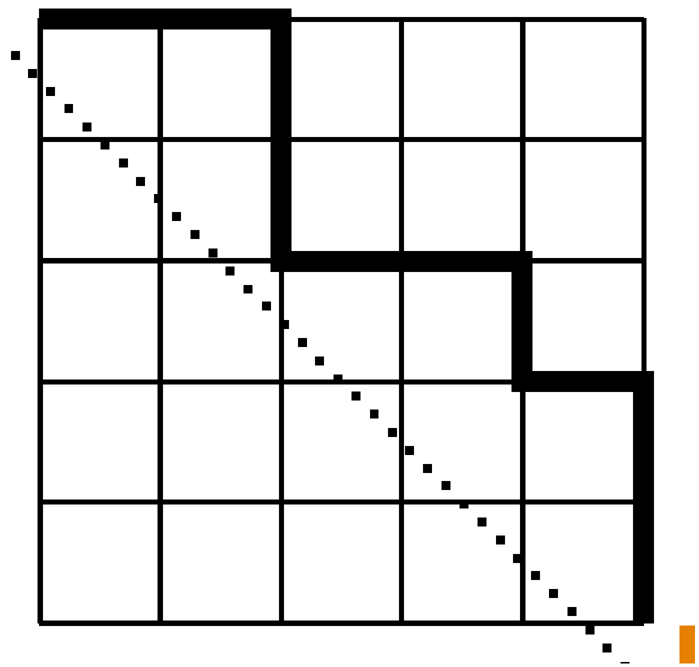
$n=0$:	`*`	1 way
$n=1$:	`()`	1 way
$n=2$:	`()(), (())`	2 ways
$n=3$:	`()()(), ()(()), (())(), (()()), ((()))`	5 ways

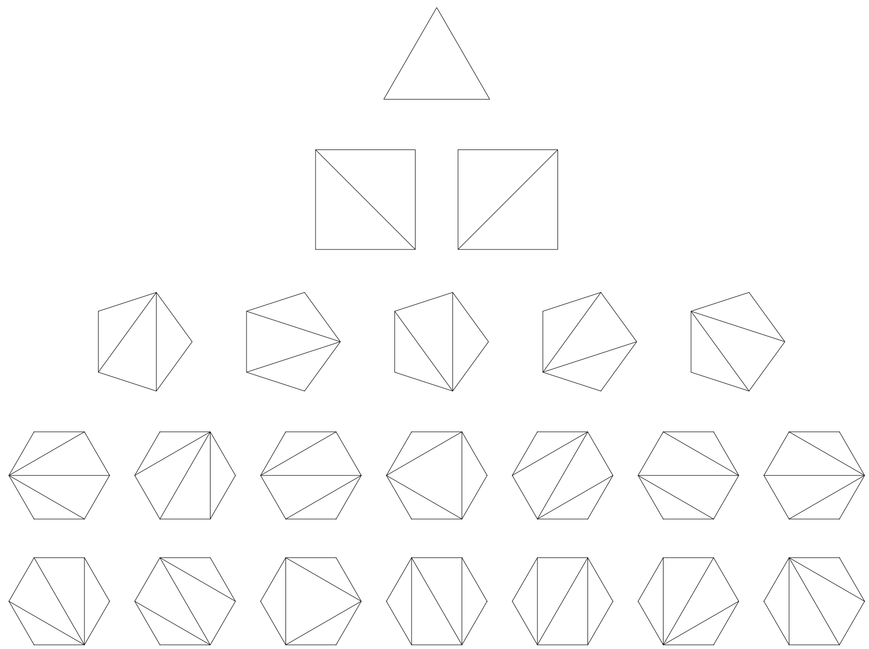
$n=4$:	`()()()(), ()()(()), ()(())(), ()(()()), ()((())),`
	`(())()(), (())(()), (()())(), ((()))(), (()()()),`
	`(()(())), ((())()), ((()())), (((())))`
		14 ways

- Number of mountains profiles with n upstrokes and n downstrokes

<table>
<tr><td>$n=0$:</td><td><pre>*</pre></td><td>1 way</td></tr>
<tr><td>$n=1$:</td><td><pre>/\</pre></td><td>1 way</td></tr>
<tr><td>$n=2$:</td><td><pre>   /\
/\/\, / \</pre></td><td>2 ways</td></tr>
<tr><td>$n=3$:</td><td><pre>   /\
           /\   /\   /\/\   / \
/\/\/\, /\/ \, / \/\, /   \, /   \</pre></td><td>5 ways</td></tr>
</table>

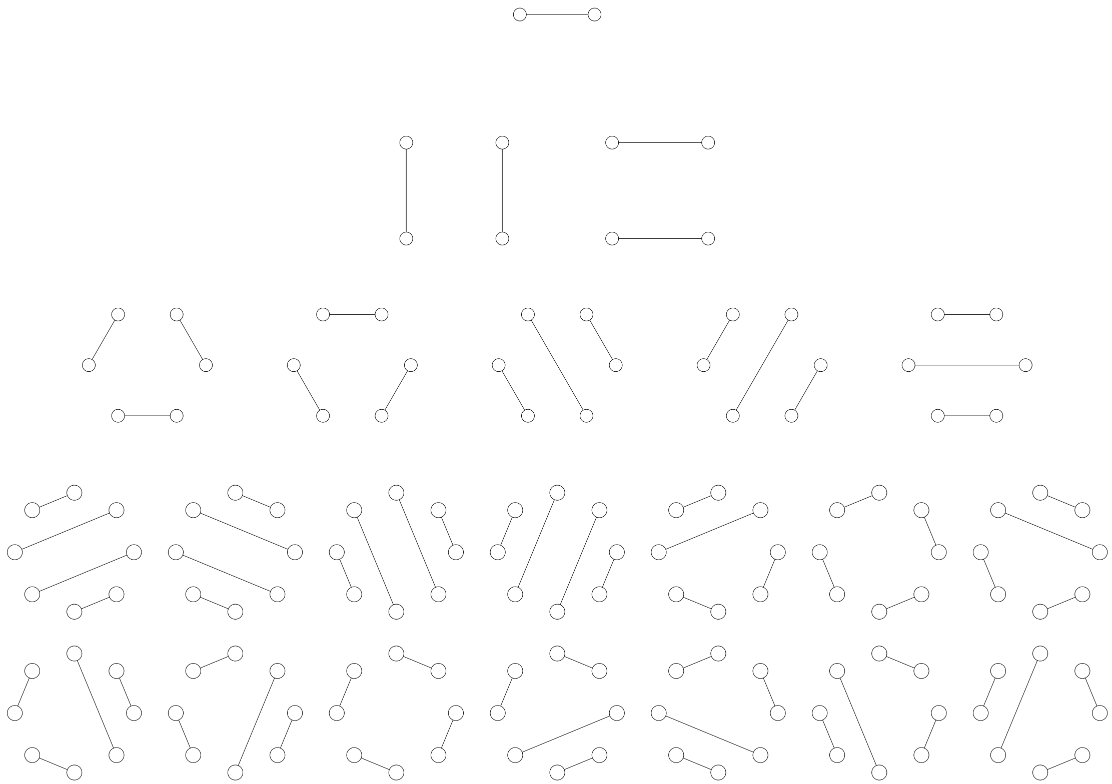
- Number of paths above (or on) the diagonal in a $n \times n$ lattice





- Number of triangulations of polygons with $n+2$ sides

- Non-crossing hand-shaking configurations of $2n$ people in a round table



- Many more!

Stanley *Enumerative Combinatorics 1999*

Normalization of the size distribution

- $P\{S=s\}$ is normalized for $p \leq 1/2$ but not for $p > 1/2$

$$\sum_{s=1}^{\infty} P\{S=s\} = \frac{q}{p} \sum_{s=1}^{\infty} C_s p^s q^s = \frac{q}{p}[h(pq)-1]$$

with $q = 1-p$ and introducing $h(x) = \sum_{s=0}^{\infty} C_s x^s$. As $h(x) = (1-\sqrt{1-4x})/(2x)$

$$h(pq) = \frac{1-\sqrt{1-4pq}}{2pq} = \frac{1-\sqrt{(1-2p)^2}}{2pq} = \frac{1-|1-2p|}{2pq} = \begin{cases} \frac{2p}{2pq} = \frac{1}{q} & \text{if } 1 \geq 2p \\ \frac{2(1-p)}{2pq} = \frac{1}{p} & \text{if } 1 \leq 2p \end{cases}$$

Therefore

$$\sum_{s=1}^{\infty} P\{S=s\} = \frac{q}{p}[h(pq)-1] = \begin{cases} \frac{q}{p}(\frac{1}{q}-1) = 1 & \text{if } p \leq 1/2 \\ \frac{q}{p}(\frac{1}{p}-1) = (q/p)^2 & \text{if } p \geq 1/2 \end{cases}$$

which turns out into

$$\sum_{s=1}^{\infty} P\{S=s\} = P_{extinction}$$

But how does $P\{S=s\}$ look like?

And what this has to do with power laws?

- Summarizing, the size distribution

$$P\{S = s\} = C_s p^{s-1}(1-p)^{s+1} = \frac{1}{s+1}\left(\begin{array}{c} 2s \\ s \end{array}\right) p^{s-1}(1-p)^{s+1}$$

for a branching process with binomial distribution and $n = 2$

- But what this has to do with power laws??

Asymptotic total size of the population

- Using Stirling's approximation, valid for $s \to \infty$ Christensen & Moloney 2005; A.C. & Font-Clos 2013

$$s! \sim \sqrt{2\pi s}\left(\frac{s}{e}\right)^s$$

the binomial coefficient turns out to be

$$\binom{2s}{s} = \frac{(2s)!}{s!s!} \sim \frac{4\pi s}{2\pi s}\frac{(2s)^{2s}}{s^{2s}} \sim \frac{4^s}{\sqrt{\pi s}}$$

and the Catalan number, replacing $s+1 \sim s$

$$C_s = \frac{1}{s+1}\binom{2s}{s} \sim \frac{4^s}{\sqrt{\pi}\, s^{3/2}}$$

essentially, an exponential increasing function of s

- Introducing the factor $p^{s-1}q^{s+1}$ we get $P\{S=s\}$

$$P\{S=s\} \sim \frac{q}{\sqrt{\pi}p}\frac{(4pq)^s}{s^{3/2}}$$

How does this function looks like for large s?

⋆ If $p(1-p) < 1/4 \Rightarrow p \neq 1/2 \Rightarrow$ decreasing exponential
⋆ If $p(1-p) = 1/4 \Rightarrow p = 1/2 \Rightarrow$ exponential dissapears $\Rightarrow$ power law!

It becomes more transparent writting

$$(4pq)^s = e^{s\ln[4p(1-p)]} = e^{-s/\xi(p)}$$

with the characteristic size defined as

$$\xi(p) = \frac{-1}{\ln[4p(1-p)]} = \left(\ln\frac{1}{4p(1-p)}\right)^{-1}$$

and then

$$P(S = s) \sim \frac{q}{\sqrt{\pi} p} \frac{e^{-s/\xi(p)}}{s^{3/2}}$$

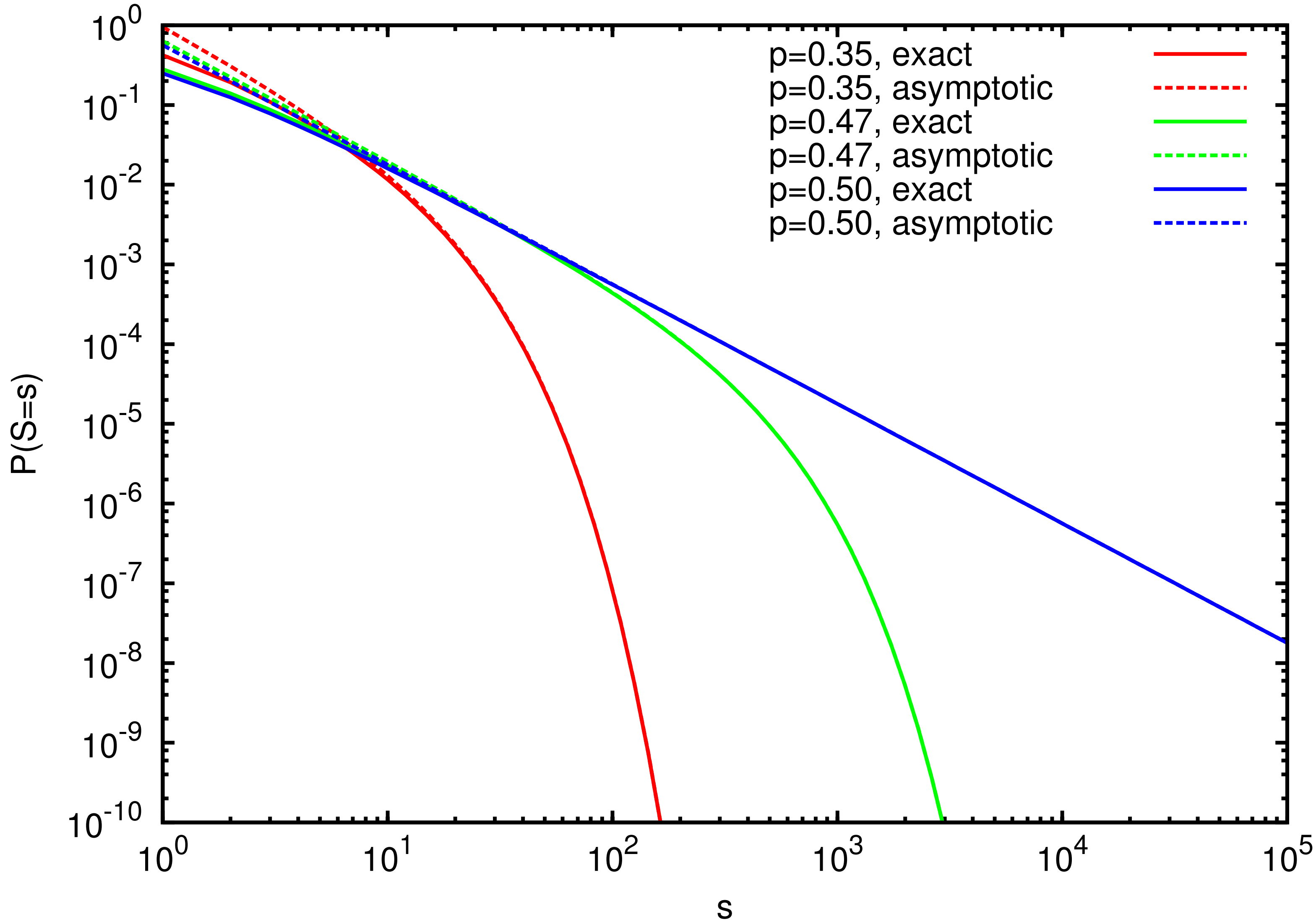
Case $p \neq 1/2$

- ⋆ For s large but $s \ll \xi(p) \Rightarrow$ power law with exponent $3/2$
- ⋆ For s large with $s \gg \xi(p) \Rightarrow$ exponential decay

Case $p = 1/2$

- ⋆ Then, $\xi \to \infty$ and for large s we obtain a power law

The critical exponent for the size distribution is 3/2



p=0.35, exact
p=0.35, asymptotic
p=0.47, exact
p=0.47, asymptotic
p=0.50, exact
p=0.50, asymptotic
P(S=s)
s

Divergence of the characteristic size

- Another critical exponent arises for the divergence of $\xi(p)$ at the critical point.

  Introducing the deviation with respect to the critical point, $\Delta \equiv p - p_c = p - 1/2$

$$p(1-p) = \left(\frac{1}{2} + \Delta\right)\left(\frac{1}{2} - \Delta\right) = \frac{1}{4} - \Delta^2$$

  So, close to the critical point (for small Δ)

$$\frac{1}{4p(1-p)} = \frac{1}{1-4\Delta^2} \simeq 1 + 4\Delta^2 + \ldots$$

  (using the formula of the geometric series), then

$$\ln \frac{1}{4p(1-p)} \simeq \ln(1+4\Delta^2) \simeq 4\Delta^2 + \ldots$$

(using the Taylor expansion of the logarithm at point 1), therefore

$$\xi(p) = \left(\ln \frac{1}{4p(1-p)} \right)^{-1} \simeq \frac{1}{4\Delta^2} + \dots$$

So, $\xi(p)$ diverges at the critical point as a power law, with an exponent $= 2$

Then, for s large and Δ small

$$P(S = s) \sim \frac{1}{\sqrt{\pi}} \frac{e^{-4(p-p_c)^2 s}}{s^{3/2}}$$

Expected value of the size

- We already know that for $\langle K \rangle < 1$ (i.e., $p < 1/2$, i.e., $\Delta < 0$)

$$\langle S \rangle = \langle N_0 \rangle + \langle N_1 \rangle + \langle N_2 \rangle + \ldots = 1 + \langle K \rangle + \langle K \rangle^2 + \ldots = \frac{1}{1 - \langle K \rangle} = -\frac{1}{2\Delta}$$

substituting $\langle K \rangle = 2p$ and $\Delta = p - 1/2 =$ deviation with respect criticality

This defines another critical exponent

- As $\xi(p) \simeq \Delta^{-2}/4$ close but below $p_c = 1/2$ then

$$\xi(p) \simeq \langle S \rangle^2$$

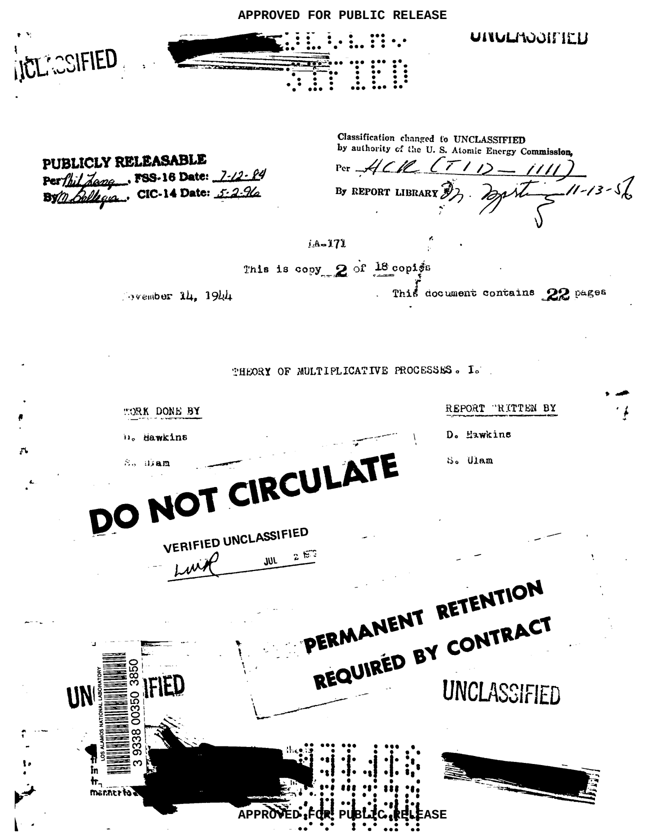
So, if the mean increases by 2, the extreme values given by ξ increase by 4

Total size of the population: general case

- Let $g(z) = f_S(z)$ be the generating function of $S = \sum_{t=0}^{\infty} N_t$. Then

$$g(z) = z f_K(g(z))$$

  with $f_K(z)$ the p.g.f. of the number of offsprings per element
- 1st demonstrated by Hawkins and Ulam in 1944 for nuclear chain reactions (as a part of the Manhattan project)
  - ⋆ A neutron may produce a fission reaction
  - ⋆ Each reaction releases neutrons
  - ⋆ Each neutron may trigger more reactions, and so on.



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THEORY OF MULTIPLICATIVE PROCESSES. I.

WORK DONE BY	REPORT WRITTEN BY
D. Hawkins	D. Hawkins
S. Ulam	S. Ulam

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- Demonstration Hawkins & Ulam 1944; A.C. & Font-Clos, arXiv 2012

Consider the size from generation 1 to ∞ (excluding the $0-$th generation)

$$S_{\tilde{0}} = S - 1 = \sum_{t=1}^{\infty} N_t$$

with $q_s = P(S_{\tilde{0}} = s)$ and a generating function $\tilde{g}(z) = \sum_{\forall s} q_s z^s$

A size s in generations from 1 to ∞ can be decomposed into

⋆ a size k in the first generation, with probability p_k, and

⋆ a size $s-k$ in the remaining generations (from 2 to ∞) but starting with k elements; this has a probability $q_{s-k}^{(k)}$ (note that $q_s = q_s^{(1)}$)

Using the law of total probability,

$$q_s = \sum_{k=1}^{s} p_k q_{s-k}^{(k)},$$

except for $s = 0$, where $q_0 = p_0$

If we multiply by z^s and sum for all s we will obtain the g.f. of $S_{\tilde{0}}$

$$\tilde{g}(z) = p_0 + \sum_{s=1}^{\infty} \sum_{k=1}^{s} p_k q_{s-k}^{(k)} z^s = p_0 + \sum_{k=1}^{\infty} p_k \left[\sum_{s=k}^{\infty} q_{s-k}^{(k)} z^{s-k} \right] z^k$$

The term in [ ] is the g.f. of the size from 1 to ∞ generations
but starting with k elements ($N_1 = k$)
As these k parents are independent of each other
$\Rightarrow$ size will be the sum of k independent random variables each with g.f. $\tilde{g}(z)$
This yields $[\tilde{g}(z)]^k$ as the corresponding generating function,

$$[\tilde{g}(z)]^k = \sum_{s-k=0}^{\infty} q_{s-k}^{(k)} z^{s-k},$$

Substituting into the equation above

$$\tilde{g}(z) = p_0 + \sum_{k=1}^{\infty} p_k [\tilde{g}(z)]^k z^k = f_K(z\tilde{g}(z))$$

As $S = 1 + S_{\tilde{0}}$ we need to add an independent variable with g.f. $= z$
(as N_0 takes the value 1 with probability 1) then, the g.f. of the size from generation 0 to ∞ is the product $z\tilde{g}$, so

$$g(z) = z\tilde{g}(z) = zf_K(z\tilde{g}(z)) = zf_K(g(z))$$

- Binomial case. Substituting $f_K(z) = (q + pz)^2$ then

$$g(z) = zf_K(g(z)) = z(q + pg(z))^2 \qquad \Rightarrow \qquad g(z) = \frac{1 - 2pqz \pm \sqrt{1 - 4pqz}}{2p^2 z}$$

Using the Taylor expansion for the square root

$$\sqrt{1 - 4pqz} = 1 - 2pqz - \sum_{s=1}^{\infty} \frac{(2s-1)!!2^{s+1}}{(s+1)!}(pqz)^{s+1}$$

where we do not need to compute the Catalan numbers C_s, so, taking "'-"

$$g(z) = \frac{q}{p} \sum_{s=1}^{\infty} C_s (pqz)^s$$

From the coefficients we recover the probability distribution we knew

$$P\{S = s\} = C_s p^{s-1} q^{s+1}$$

- Geometric case. "Success" probability p and $q = 1 - p$ and values $k = 0, 1, 2 \ldots \infty$

$$p_k = P\{K = k\} = q^k p \qquad \Rightarrow \qquad f_K(z) = \sum_{k=0}^{\infty} p_k z^k = p \sum_{k=0}^{\infty} q^k z^k = \frac{p}{1 - qz}$$

using the geometric series. The generating function for the size is

$$g(z) = z f_K(g(z)) = \frac{pz}{1 - qg(z)} \quad \Rightarrow \quad g(z) = \frac{1 - \sqrt{1 - 4pqz}}{2q} = pz + \sum_{s=2}^{\infty} C_{s-1} q^{s-1} p^s z^s$$

where we have used the following, with C_i the $i-$th Catalan number

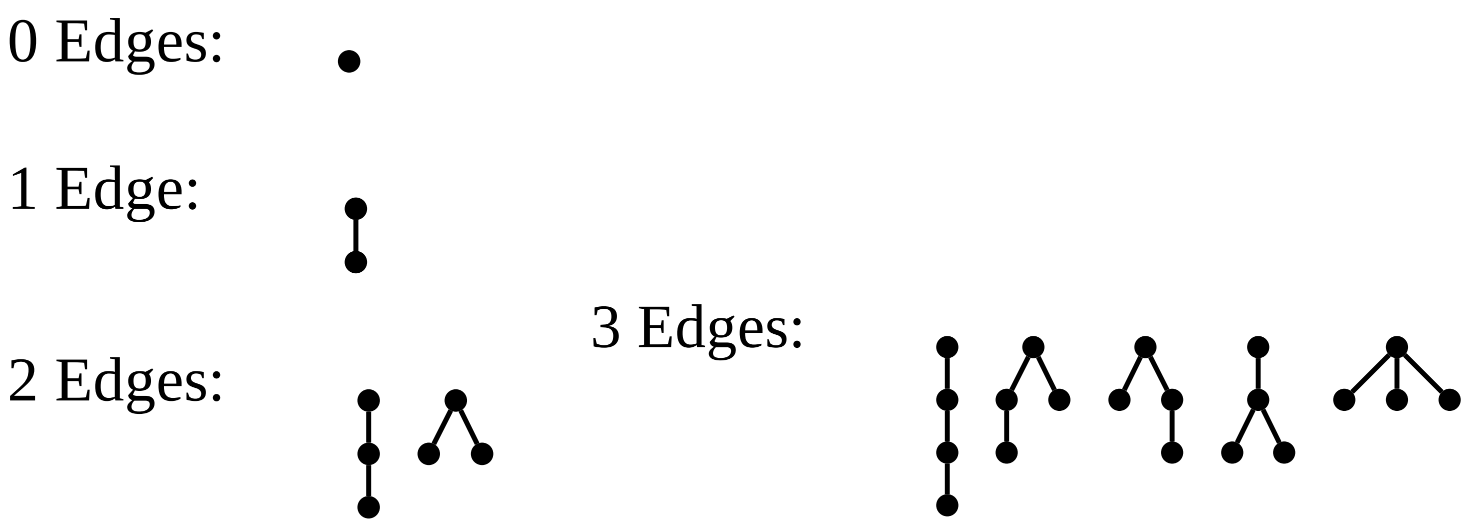
$$\sqrt{1 - 4x} = 1 - 2x - \sum_{i=1}^{\infty} 2C_i x^{i+1}$$

Therefore, the size distribution (without binary trees!)

$$P\{S = s\} = C_{s-1} q^{s-1} p^{s} \qquad \sim \frac{1}{4\sqrt{\pi} q} \frac{(4pq)^{s}}{s^{3/2}} \text{ for } s \to \infty$$

so we again obtain a critical exponent $= 3/2$ (and also the others)

- C_s also counts number of (non-necessarily-binary) trees with s edges



Davis 2010

- Normalization of the size distribution in the geometric case

$$\sum_{s=1}^{\infty} P\{S = s\} = C_{s-1} q^{s-1} p^{s} = \begin{cases} 1 & \text{if } q \leq 1/2 \\ p/q & \text{if } q \geq 1/2 \end{cases}$$

which corresponds to the probability of extinction in the geometric case
Note that $\langle K \rangle = q/p$, so $p_C = q_C = 1/2$

- Another offspring distribution
  - 0 offsprings with probability $q = 1 - p$
  - 2 offsprings with probability p

  Then $f_K(z) = q + pz^2$. The generating function for the size is

$$g(z) = z f_K(g(z)) = z(q + pg(z)^2) \quad \Rightarrow \quad g(z) = \frac{1 \pm \sqrt{1 - 4pqz^2}}{2pz} = \sum_{i=0}^{\infty} C_i p^i q^{i+1} z^{2i+1}$$

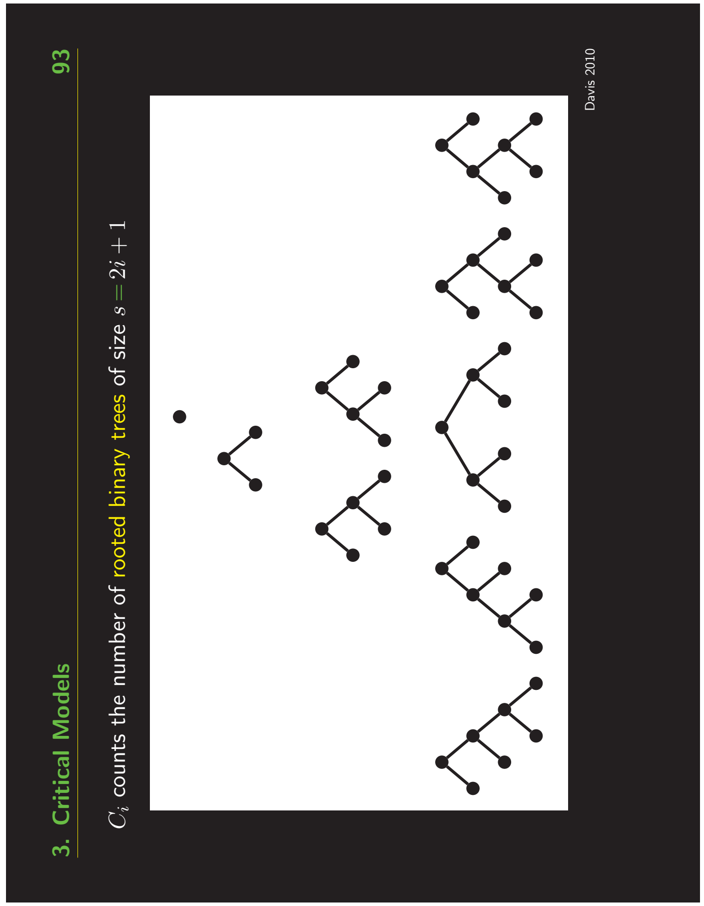
Therefore

$$P\{S = s\} = C_{\frac{s-1}{2}} \, p^{\frac{s-1}{2}} q^{\frac{s+1}{2}} \qquad \text{for } s = 1, 3, 5 \ldots$$

So, C_i counts the number of rooted binary trees of size $s = 2i + 1$
Asymptotically we do not scape from the exponent 3/2

$$P\{S = s\} \sim \sqrt{\frac{2q}{\pi p}} \frac{(4pq)^{s/2}}{s^{3/2}} \; \text{ for } s \to \infty$$

C_i counts the number of rooted binary trees of size $s = 2i + 1$



Finite size effects in branching processes

Garcia-Millan, Font-Clos & A. C. *Phys Rev E* 2015

- Let us consider a limitation in the number of generations: $t = 0, 1, \ldots L$ (this plays the role of boundaries)

  The probability of extinction, with $f(z) \equiv f_K(z)$, will be

$$P_{ext}(L) = f^L(0) \qquad < \qquad P_\infty = \lim_{t \to \infty} f^t(0)$$

- Consider a very large number of generations, n

  $\Rightarrow$ $f^n(0)$ will be close to $f^\infty(0) = P_\infty$

  Let us Taylor expand $f(f^n(0))$ around the fixed point P_∞

$$f^{n+1}(0) = f(f^n(0)) = P_\infty + f'(P_\infty)\,(f^n(0) - P_\infty) + \ldots$$

- Taking up to 2nd-order terms and arranging, the inverse of the distance is[1]

$$c_{n+1} \equiv \frac{1}{P_\infty - f^{n+1}(0)} = \frac{c_n}{M} + \frac{C}{M^2}$$

with $M = f'(P_\infty)$ and $C = f''(P_\infty)/2$. Iterating

$$c_{n+\ell} = \frac{c_n}{M^\ell} + \frac{C(1-M^\ell)}{M^{\ell+1}(1-M)}$$

In the subcritical case, $P_\infty = 1$, then $M = \langle K \rangle$ and $2C = \sigma^2 + \langle K \rangle(\langle K \rangle - 1)$, so

$$c_{n+\ell} = \frac{c_n}{\langle K \rangle^\ell} + \frac{\sigma^2(1-\langle K \rangle^\ell)}{2\langle K \rangle^{\ell+1}(1-\langle K \rangle)} - \frac{1-\langle K \rangle^\ell}{2\langle K \rangle^\ell}$$

[1] do not confuse distance to the fixed point with distance to the critical point

- Let us introduce a rescaled distance to the critical point $y = \ell(\langle K \rangle - 1)$, so

$$c_{n+\ell} = \frac{\sigma^2(1 - \langle K \rangle^\ell)}{2\langle K \rangle^{\ell+1}(1 - \langle K \rangle)} + \ldots = -\frac{\sigma_c^2(1 - e^y)\ell}{2e^y y}$$

with $\langle K \rangle = 1 + y/\ell$ and $\langle K \rangle^\ell = e^y$ and with ℓ large (then $\langle K \rangle$ is close to 1)

For $L = \ell + n \gg n$, we have that the probability of non-extinction will be

$$\rho(L) = 1 - P_{ext}(L) = 1 - f^L(0) = \frac{1}{c_L} = \frac{2e^y y}{\sigma_c^2(e^y - 1)L},$$

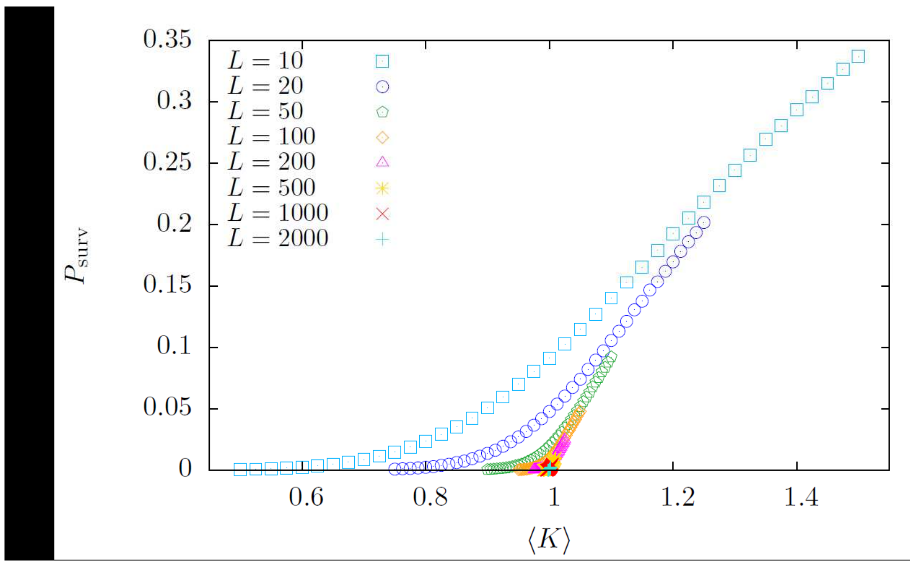
with $L \simeq \ell$. So, a scaling law is fulfilled, with scaling function $\mathcal{G}(y)$

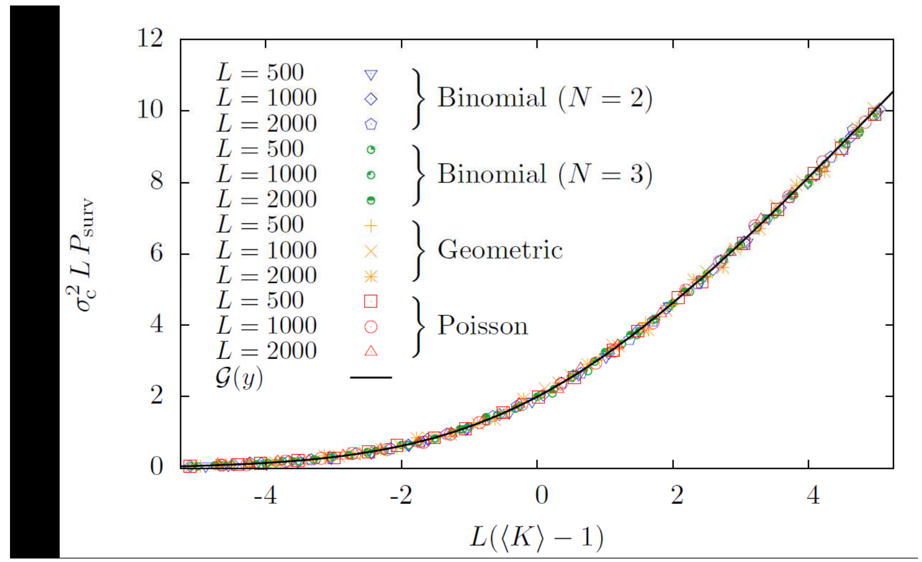
$$\rho(L) = \frac{1}{L\sigma_c^2}\mathcal{G}(L(\langle K \rangle - 1)) \quad \text{with} \quad \mathcal{G}(y) = \frac{2ye^y}{e^y - 1}$$

valid also for the supercritical case. This is called finite-size scaling

- Let us repeat

$$\rho(L) = \frac{1}{L\sigma_c^2}\mathcal{G}(L(\langle K\rangle - 1)) \quad \text{with} \quad \mathcal{G}(y) = \frac{2ye^y}{e^y - 1}$$



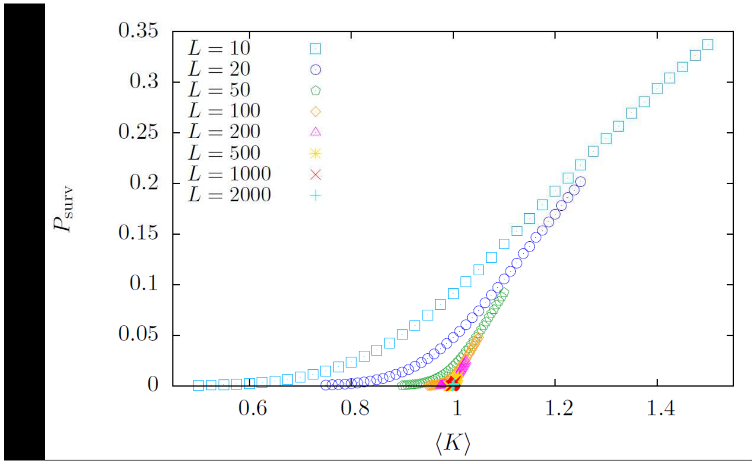


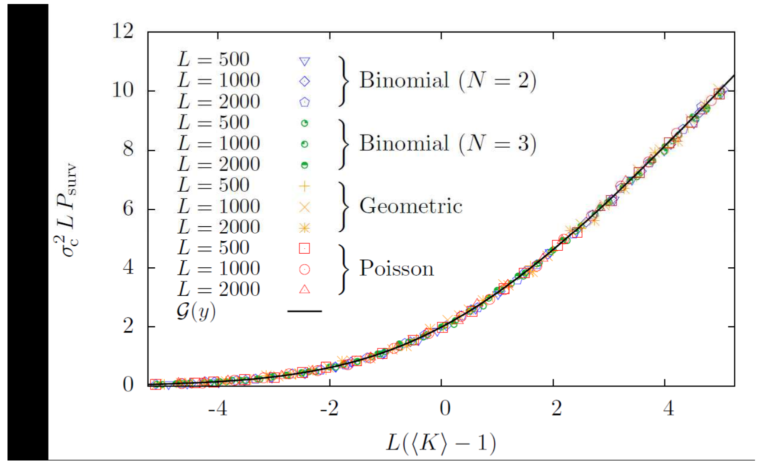
Phase transitions only exist in the infinite-system limit (thermodynamic limit)

$$\mathcal{G}(y) = \frac{2ye^y}{e^y - 1} \quad \rightarrow \quad \begin{cases} 0 & \text{when } y \rightarrow -\infty \\ 2 & \text{when } y \rightarrow 0 \\ 2y & \text{when } y \rightarrow \infty \end{cases}$$

So, for $L \rightarrow \infty$

$$\rho(L) = \frac{1}{L\sigma_c^2}\mathcal{G}(L(\langle K \rangle - 1)) \quad \rightarrow \quad \begin{cases} 0 & \text{for } \langle K \rangle < 1 \\ 2\sigma_c^{-2}L^{-1} & \text{for } \langle K \rangle = 1 \\ 2(\langle K \rangle - 1)/\sigma_c^2 & \text{for } \langle K \rangle > 1 \end{cases}$$





Simulation of a branching process

- Initialize $t = 0$ and $N_0 = 1$ (one single ancestor)

- Loop for t
  - ⋆ Simulate N_t values of K_i
  - ⋆ Compute $N_{t+1} = \sum_{i=1}^{N_t} K_i$
  - ⋆ If $N_{t+1} = 0 \Rightarrow$ stop
  - ⋆ $t = t + 1$

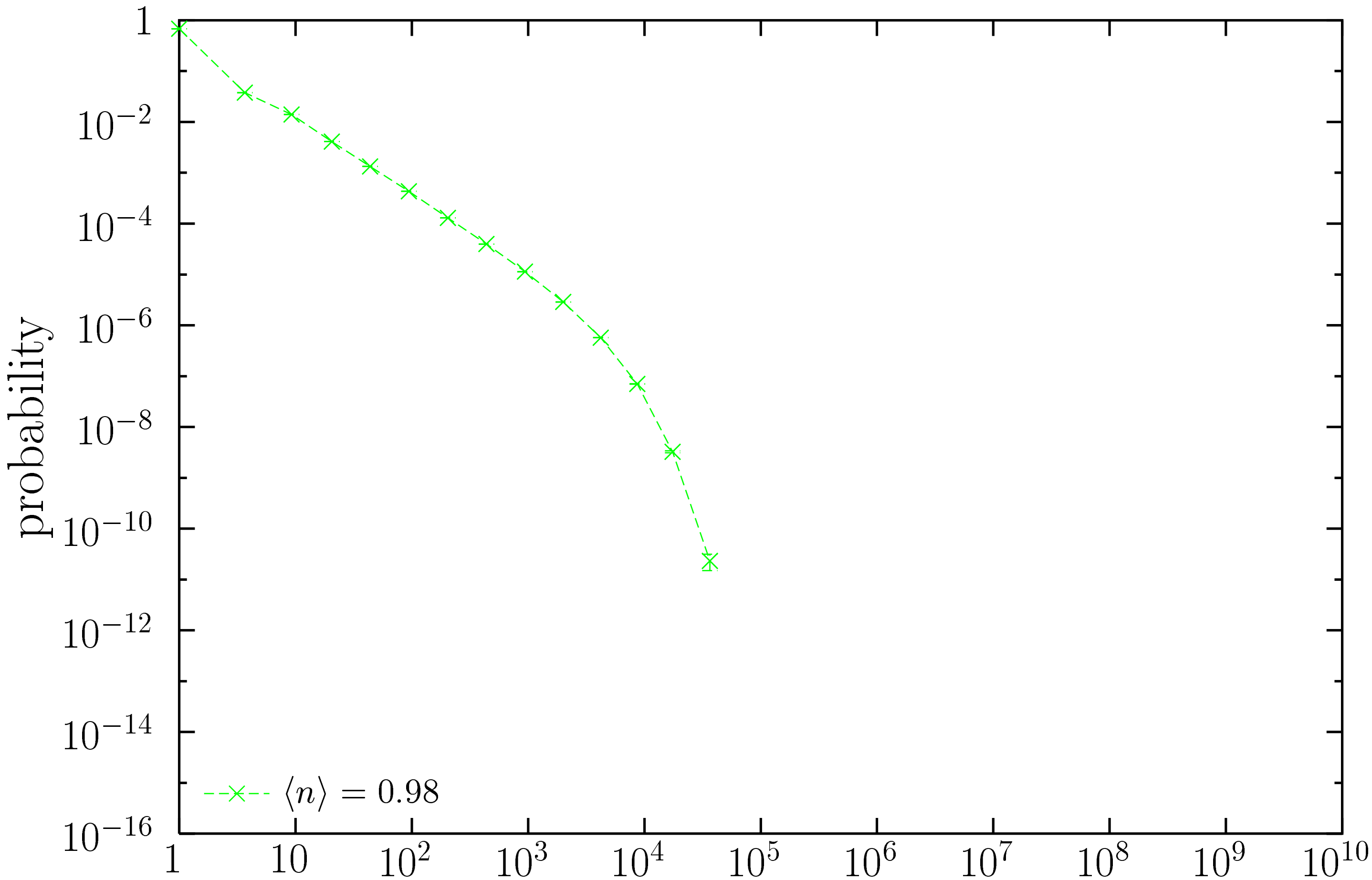
- For the twins-or-nothing example

$$K = \begin{cases} 2 & \text{if } u \leq p \\ 0 & \text{otherwise} \end{cases}$$

with u a uniform random number between 0 and 1

As the mean of the number of offsprings is $\langle K \rangle = 2p$, then, $p_c = 1/2$

Plot of $P\{S = s\}$ (with $\langle n \rangle = \langle K \rangle$)



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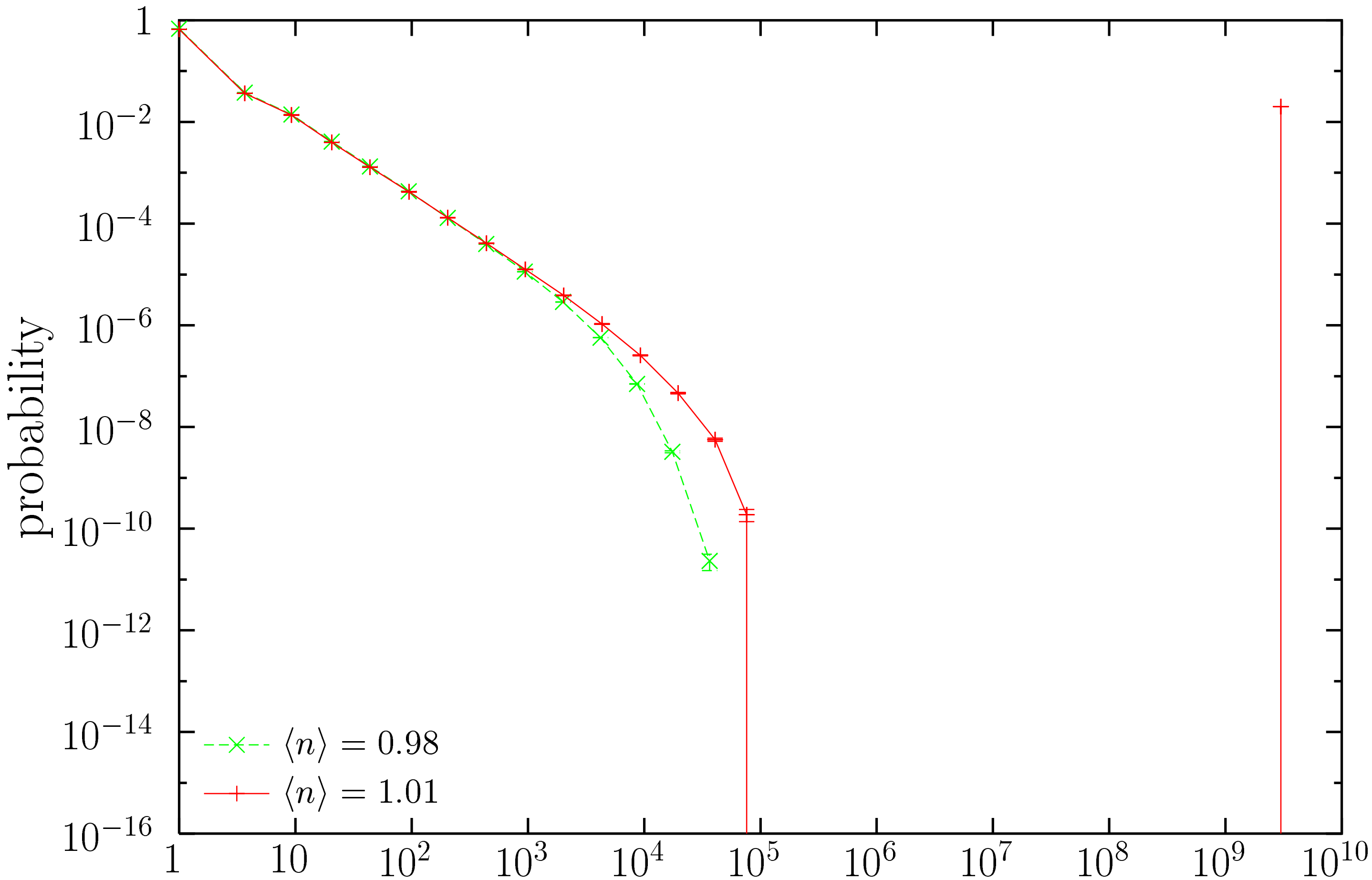

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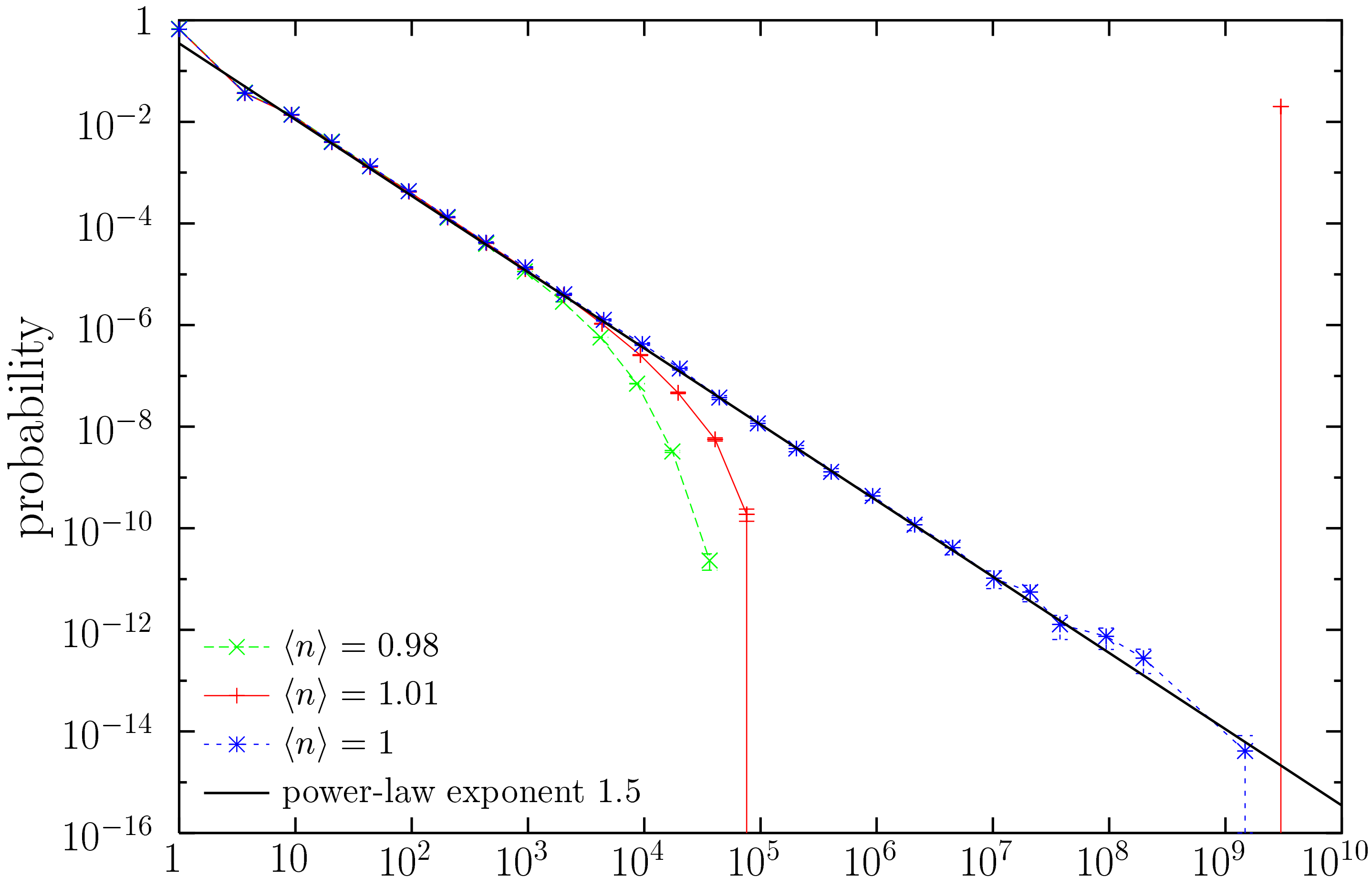


Earthquakes and branching processes

- Gutenberg-Richter power law holds only for $\langle K \rangle = 1$
  Critical branching process $\Rightarrow$ Fine tuning of mean number of offsprings
  $\Rightarrow$ Very difficult to get in practice!

- Agreement only qualitative, not quantitative

$$1 + \frac{2b}{3} \simeq 1.67 \qquad \neq \qquad \frac{3}{2}$$

- Model too simple, still

- Kagan: Gutenberg-Richter exponent should be 3/2 (i.e., $b-$value $= 0.75$)

  Instrumental artifacts makes the exponent increase Kagan, *Tectonophys* 2010

Consequences for predictability

- Consider $\langle N_{t+1}|N_t\rangle$ with N_t known, then

$$\langle N_{t+1}|N_t\rangle = \langle K\rangle N_t$$

  using $N_{t+1} = \sum_{i=1}^{N_t} K_i$

  For critical branching processes $\langle K\rangle = 1$ and then

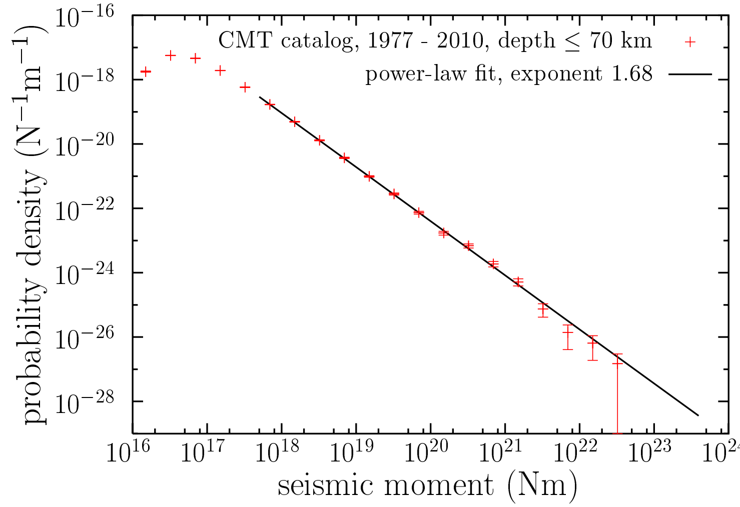
$$\langle N_{t+1}|N_t\rangle = N_t$$

  Note that it is not only that the outcome of the next step is random

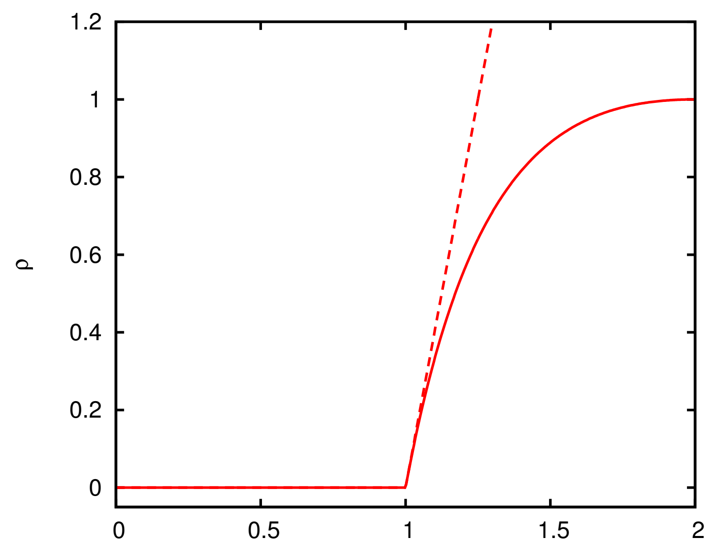
  It is much worst: the earthquake is in the limit of attenuation and intensification

- But what makes earthquakes critical?

Summary



- The size (energy) of earthquakes (and other natural hazards) follows a power-law distribution
- A power law signals the absence of a characteristic scale
- (Decreasing) power-law densities, with $\beta \leq 2$ have an infinite mean value
- Galton-Watson branching process can be a model of earthquakes
  - ⋆ Continuous phase transition at $\langle K \rangle = 1$
  - ⋆ Size distribution is only power law at the critical point



Self-Organized Branching Process

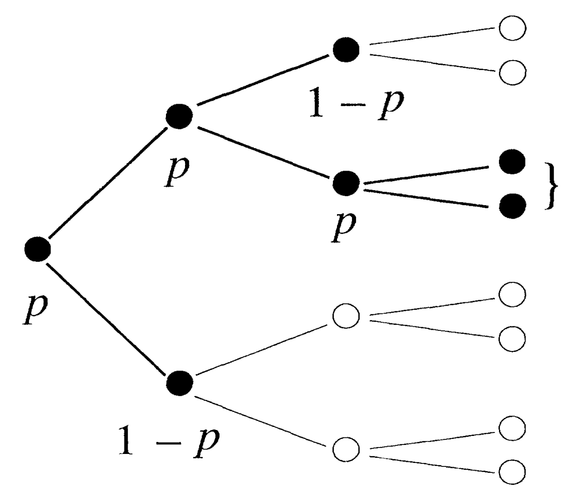
Zapperi et al. *Phys Rev Lett* 1995

- Consider: 0 offsprings with prob $1-p$
  2 offsprings with prob p

  Limit the maximum number of generations
  $\Rightarrow$ analogous to introduce a boundary at $t=L$

  Change p from one realization T to the next as

$$p(T+1)=p(T)+\frac{1-N_L(T)}{M}$$

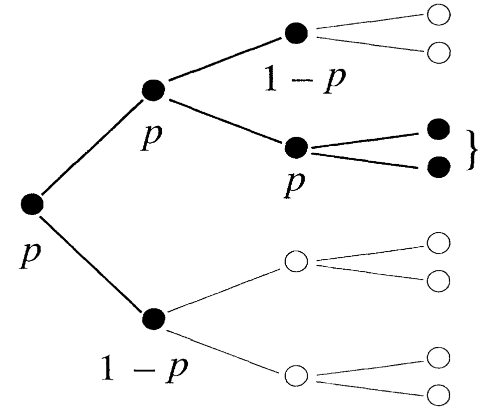


where N_L is the population in the last generation ($=2$ in Fig.)

and M is a big number (explained later)

Note that there are 2 times scales

⋆ $t=$ fast time scale, counts generations ,from $t=0$ to L
⋆ $T=$ slow time scale, counts realizations



$$p(T+1) = p(T) + \frac{1 - N_L(T)}{M}$$

- Dynamics

  ⋆ If p is low $\Rightarrow$ small size $\Rightarrow$ $N_L = 0$ $\Rightarrow$ p increases
  ⋆ If p is high $\Rightarrow$ large size $\Rightarrow$ $N_L > 1$ $\Rightarrow$ p decreases

- Indeed, we know that $\langle N_L \rangle = \langle K \rangle^L = (2p)^L$

  So, we can write, $N_L = (2p)^L + \eta$, with $\langle \eta \rangle = 0$

  Considering the deterministic equation (removing η)

$$p(T+1) = F(p(T)) = p(T) + \frac{1 - (2p(T))^L}{M}$$

  Therefore, the deterministic equation has a fixed point $p^* = 1/2 = p_c$

- Moreover, if M is big enough then $|F'(p^*)| < 1$ and the fixed point is attractive,so

$$p(T) \to p^* = p_c$$

As the noise is small, it only adds small perturbations to p^*

Then, p tends, or self-organizes, to its critical value, on average

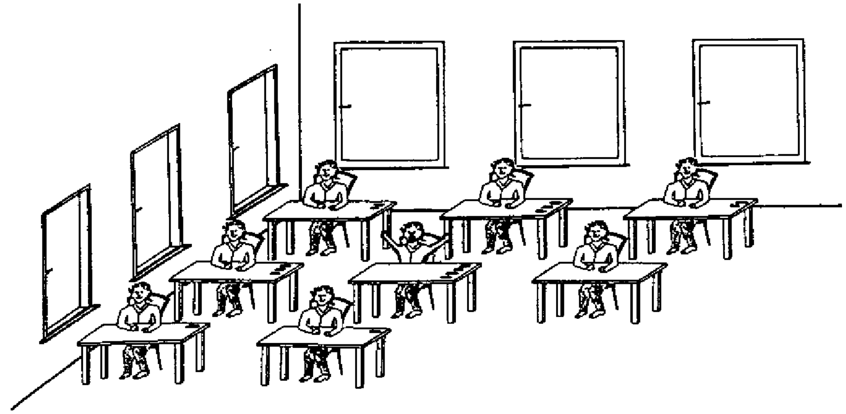
- Note:
Self-organization is the spontaneous emergence of structures or global order (here we do not have any structure yet, but wait...)

Examples:
convection patterns in fluids, chemical oscillations, self-regulations of markets

- Nevertheless, the global condition (on p) is very difficult to justify, in practice

4. Self-organization towards criticality

Cellular automaton Manna model[2]



Bak 1996, after Grassberger

- Let us consider a lattice in d dimensions
  - ⋆ Each site can store only 1 particle (or 0)
  - ⋆ If extra particles arrive at a site:
    - ⇒ 2 of them are transferred to 2 randomly chosen sites among its neighbors (this may generate an avalanche)
  - ⋆ Particles leave the system through the (open) boundary
  - ⋆ If there is no activity (all sites with 1 particle or less):
    - ⇒ Add 1 particle to a random site

In a formula, with $nn(j)$ denoting 2 random neighbors of j

$$\text{if } z_j \geq 2 \quad \Rightarrow \quad \begin{cases} z_j & \rightarrow & z_j & - & 2 \\ z_{nn(j)} & \rightarrow & z_{nn(j)} & + & 1 \end{cases}$$

$$\text{if } z_k < 2 \ \forall k \quad \Rightarrow \quad z_n \rightarrow z_n + 1 \ \text{ with } n = rand$$

[2] Cellular automaton = dynamical system with discrete time, space, and variable (field)

- The Manna model defines a complex system:

  System composed of many interacting parts, such that the collective behavior of those parts together is more than the sum of their individual behaviors

  Other examples, more complex: the cell, the brain, ecosystems, the economy, the Earth's crust... Newman, *Am J Phys* 2011

- Let us go back to the Manna model in the limit of infinite dimensions, $d \to \infty$

  Then, the propagation of the activity will show no loops $\simeq$ mean field (a neighbor will not be selected twice to get a grain $\Rightarrow$ no overlap)

  So, there will be no spatial correlations, and all sites are equivalent (the boundary conditions need to be readjusted)

  Each site will become active $(z \geq 2)$ with the same probability

$$p = \text{fraction of sites with one particle} = P[z = 1]$$

- Then, the activity propagates through the system as a branching process
- The offspring distribution will be binomial, with $n = 2$ and parameter p

  But note that there is no pre-existing tree
- The total number of particles will evolve as

$$\text{mass}(T+1) = \text{mass}(T) + 1 - \text{out}(T)$$

(one particle added before the avalanche, "out" particles lost at the boundaries)

Dividing by the total number of sites M, with $p = P[z = 1] = \text{mass}/M$

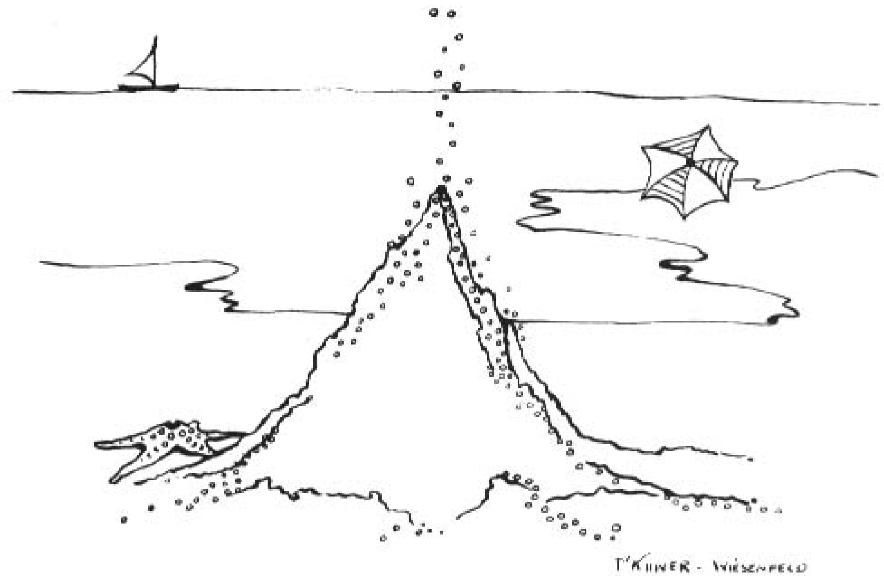
$$p(T+1) = p(T) + \frac{1 - \text{out}(T)}{M}$$

which corresponds to the self-organized branching process Christensen & Moloney 2005

The evolution and adjustment of p is implemented in a natural way

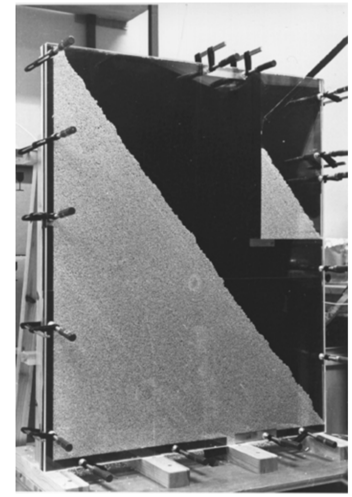
Self-Organized Criticality (SOC) Bak et al. *Phys Rev Lett* 1987

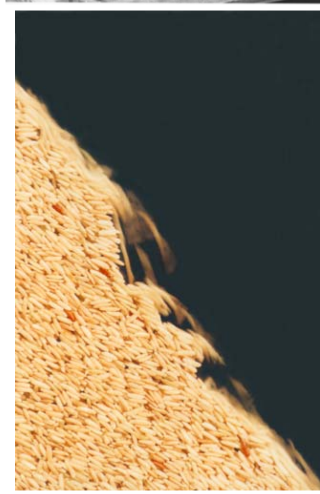
- The dynamics arises from the sandpile metaphor
  - ⋆ If there are few grains (flat pile)
    $\Rightarrow$ small avalanches, pile grows
  - ⋆ If there are many grains (steep pile)
    $\Rightarrow$ large avalanches, pile decreases
    (through boundary dissipation)



This mechanism makes the slope of the pile fluctuate around the critical state

$\Rightarrow$ Bak-Tang-Wiesenfeld (BTW) model





- BTW model: one-dimensional lattice, $d = 1$, with $j = 1 \ldots L$

  Modification: no random selection of neighbors

$$
\begin{array}{lll}
\text{if } z_j \geq 2 & \Rightarrow & \left\{ \begin{array}{lllll} z_j & \to & z_j & - & 2 \quad \text{for } j \neq L \\ z_{j\pm 1} & \to & z_{j\pm 1} & + & 1 \end{array} \right. \\
\text{if } z_k < 2 \ \forall k & \Rightarrow & z_n \to z_n + 1 \quad \text{with } n = rand
\end{array}
$$

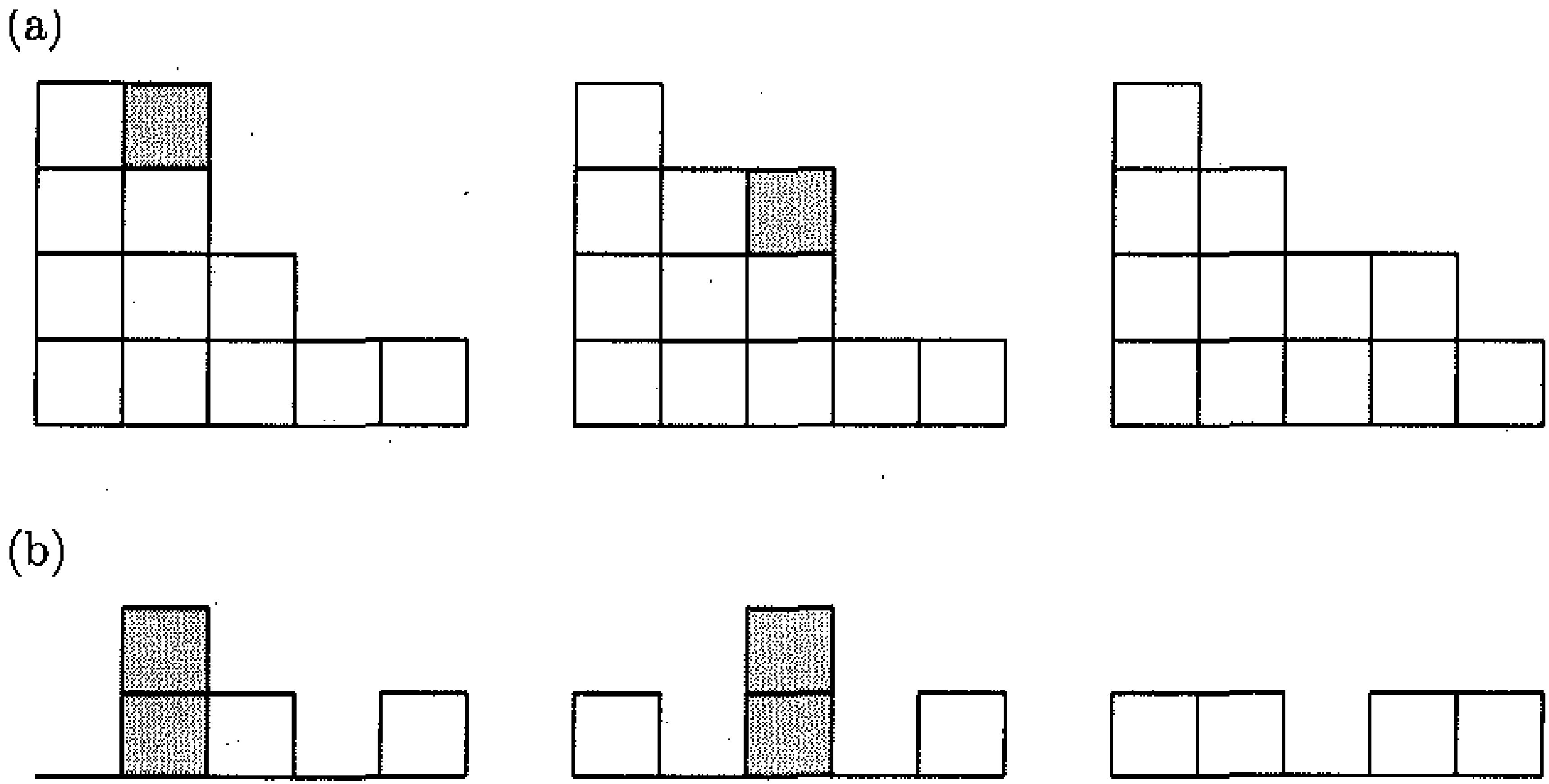
The "particles" are in fact elements of slope in a $2-d$ sandpile

$$\text{height at } j = h_j = h_{j+1} + z_j \quad \Rightarrow \quad z_j = h_j - h_{j+1}$$

with $h_{L+1} = 0 \Rightarrow z_L = h_L \Rightarrow z_L \to z_L - 1$ (conserved BC)

$$
\begin{array}{lll}
\text{if } h_j - h_{j+1} \geq 2 & \Rightarrow & \left\{ \begin{array}{lllll} h_j & \to & h_j & - & 1 \\ h_{j+1} & \to & h_{j+1} & + & 1 \end{array} \right. \\
\text{if } h_k - h_{k+1} < 2 \ \forall k & \Rightarrow & h_m \to h_m + 1 \text{ for } m \leq n \text{ with } n = rand
\end{array}
$$

- Height h picture (grains) versus slope z picture (repelling particles)



Christensen & Moloney 2005

Relation with interface depinning

Paczuski & Boettcher, *Phys Rev Lett* 1996

- Define H_j as the total number of topplins in a sandpile

  When:

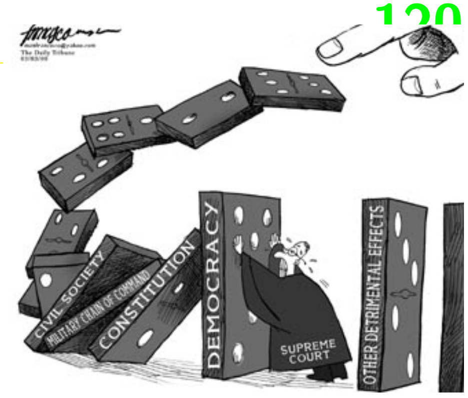
  ⋆ the initial condition is empty ($h_j = 0$ for all j and for $T = 0$) and
  ⋆ the addition takes place at $j = 1$

  then, H defines an advancing interface, whose gradient gives the pile height

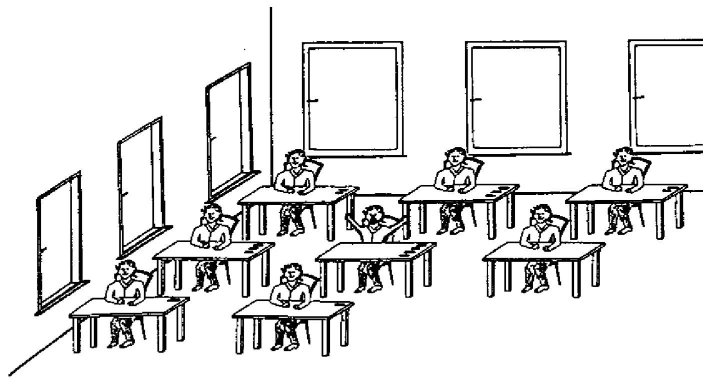
$$h_j = H_{j-1} - H_j$$

  with H_0 giving the total number of grains added

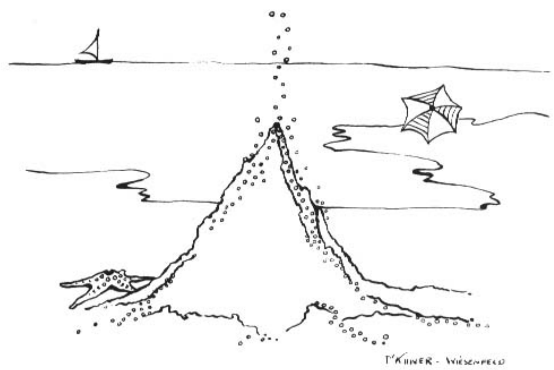
Retrospective of models



- Domino (Otsuka) model of fracture
- Galton-Watson branching process
- Self-organized branching model



- Cellular automaton Manna (bureaucrats) model
- Bak et al. sandpile model



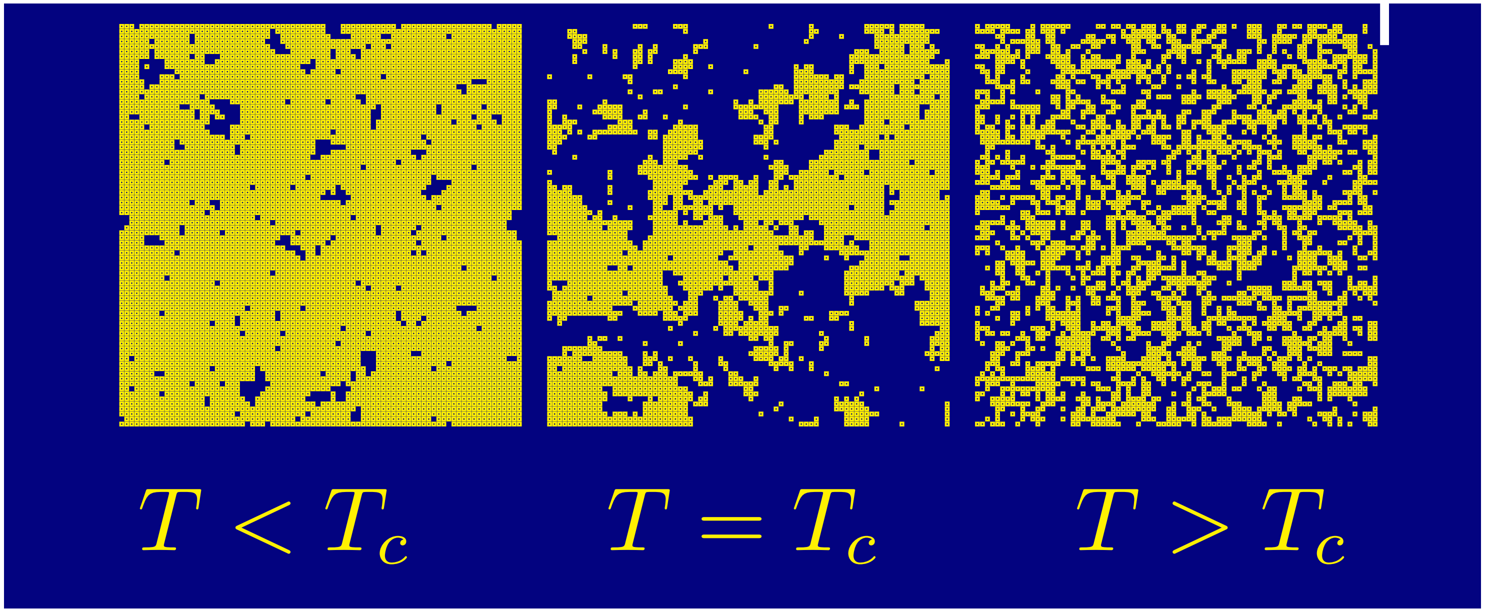
- Interface depinning model
- These models serve as metaphors for earthquakes

- **Inspiration: Critical Points of Thermodynamic Phase Transitions**

  Magnetic material: atom $=$ spin with 2 states

  There exists a critical temperature T_c

  ⋆ Above T_c: no magnetization, small clusters
  ⋆ Below T_c: magnetization, one very large cluster
  ⋆ At the precise value $T = T_c$ $\Rightarrow$ clusters of all sizes $\Rightarrow$ power law!



Christensen & Moloney, *Complexity and Criticality* 2005

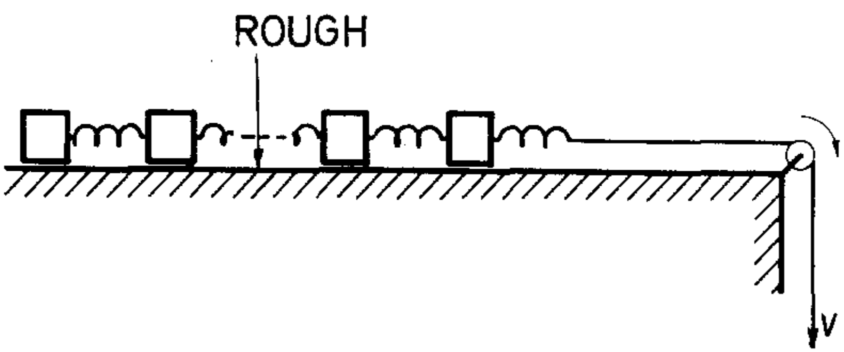
Burridge-Knopoff spring-block models

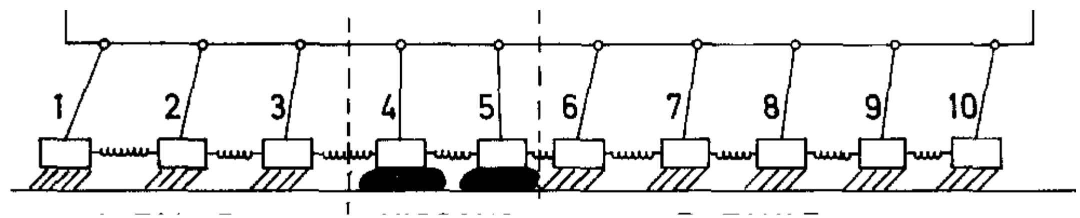
Bull Seism Soc Am 1967

- Earthquakes take place in "pre-existing" faults

⇒ Alternative: modeling friction in a fault

⋆ Experiment: spring-block system pulled from one end





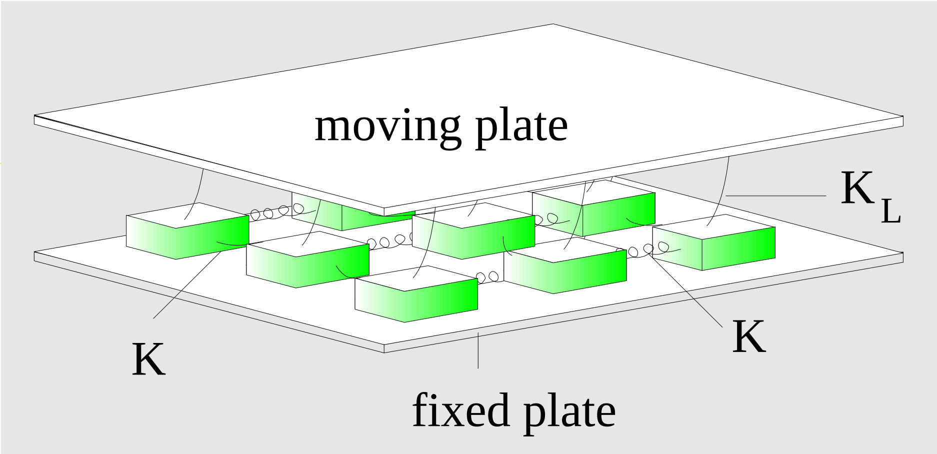
⋆ Computer simulations:
All blocks connected by flat springs to a moving plate

- stick-slip dynamics: slow driving (pull) + fast avalanches (shocks)
  - ⋆ The force on the block(s) increases (linearly) very slowly
  - ⋆ At some time (for some block) the force exceeds the static frictional force
  - ⋆ Then, that block moves fast, changing the force over the neighbor blocks and so on

  “Size” of the earthquake $\simeq$ number of sliding blocks

4. Self-organization towards criticality

Coupled-map lattice model

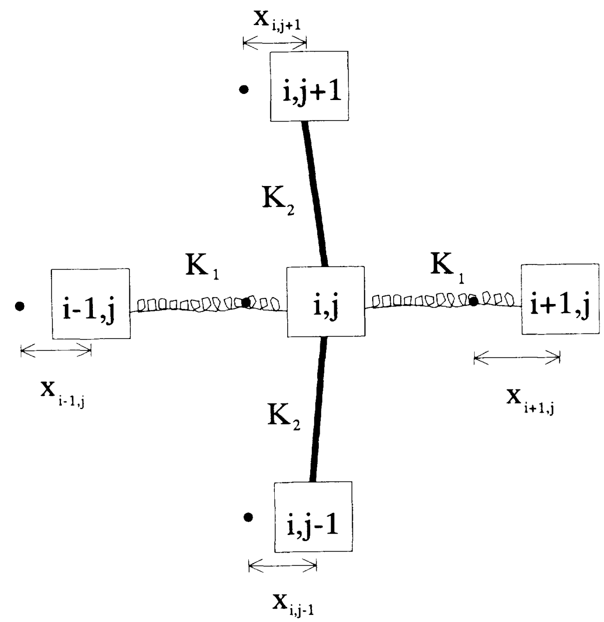


- Olami-Feder-Christensen (OFC) model, *Phys Rev Lett* 1992

  Two-dimensional version of Burridge-Knopoff model

  - ⋆ Coil (helical) springs connecting blocks in the direction of motion of the plate
  - ⋆ Flat (leaf) springs connecting blocks in the perpendicular direction (making the force then in the direction of motion also)
  - ⋆ In both cases the value of the elastic constants is K
  - ⋆ Flat springs connecting blocks with the upper moving plate with constants $K_L \neq K$

4. Self-organization towards criticality



- Let us define
  - ⋆ $F_{i,j} =$ Force on block i, j
  - ⋆ $x_{i,j} =$ Displacement in the direction of motion of i, j relative to the upper flat spring

  Also, the zero force between each pair corresponds to the lattice of upper springs. By Hooke's law $F_{i,j} =$

$$= -K(x_{i,j} - x_{i-1,j}) - K(x_{i,j} - x_{i+1,j}) - K(x_{i,j} - x_{i,j-1}) - K(x_{i,j} - x_{i,j+1}) - K_L x_{i,j}$$

$$= K \left(\sum_{nn(i,j)} x_{nn(i,j)} - 4x_{i,j} \right) - K_L x_{i,j}$$

If the upper plate moves with constant (small) velocity v then

$$\frac{dF_{k,l}}{dt} = -K_L \frac{dx_{k,l}}{dt} = K_L v \quad \text{for all } k, l$$

- When the force on some block i, j reaches the frictional threshold force F_{th} $\Rightarrow$ block i, j slips instantaneously to the position with of zero force, so

$$F_{i,j} \rightarrow 0 \qquad \text{(assumption of the model)}$$

Then, if we denote the new position of i, j as $x'_{i,j}$

$$0 = K \left(\sum_{nn(i,j)} x_{nn(i,j)} - 4x'_{i,j} \right) - K_L x'_{i,j}$$

where $nn(i, j)$ denotes the nearest neighbors of i, j. Substracting,

$$F_{i,j} - 0 = -(4K + K_L)(x_{i,j} - x'_{i,j})$$

- Therefore, the force on the $i+1, j$ neighbor (for instance)

$$F_{i+1,j} = K \left(\sum_{nn(i+1,j)} x_{nn(i+1,j)} - 4x_{i+1,j} \right) - K_L x_{i+1,j}$$

So, as $F_{i,j} = -(4K + K_L)(x_{i,j} - x'_{i,j})$ then $F_{i+1,j}$ changes to

$$F_{i+1,j} \rightarrow F_{i+1,j} + K(x'_{i,j} - x_{i,j}) = F_{i+1,j} + \frac{K}{4K + K_L} F_{i,j}$$

and the model is non-conservative, as $\alpha = K/(4K + K_L) < 0.25$ except if $K_L \rightarrow 0$

Summary of the rules of the OFC model

if $F_{i,j} < F_{th}$ for all i, j $\Rightarrow$ $dF_{i,j}/dt = K_L v$ with v very small

if $F_{i,j} \geq F_{th}$ for some i, j $\Rightarrow$ $\begin{cases} F_{nn(i,j)} & \rightarrow & F_{nn(i,j)} + \alpha F_{i,j} \\ F_{i,j} & \rightarrow & 0 \end{cases}$

The boundary conditions are disregarded

Note that there are 2 times scales:
The slow one is continuous, but the fast one is discrete

In practice, in simulations, don't use $dF_{i,j}/dt = K_L v$. Why?

Then, the slow time scale turns into discontinuous

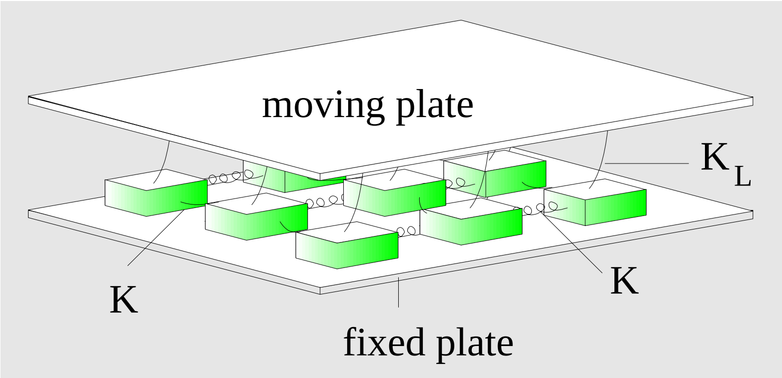
$\Rightarrow$ coupled map lattice model[3]

[3] Coupled map lattice = dynamical system on a lattice with continuous variables and discrete time

Earthquakes *can be* a SOC phenomenon

- Ingredients for SOC (and fulfillment in earthquakes)

Pruessner, private comm.



  - ⋆ Time scale separation ($\Rightarrow$ OK)
  - ⋆ Thresholds, interaction ($\Rightarrow$ OK)
  - ⋆ Avalanche dynamics ($\Rightarrow$ OK)
  - ⋆ Power-law distributions ($\Rightarrow$ OK)
    (with finite-size scaling)

  - ⋆ Underlying 2nd-order phase transition,
    reached by self-organization ($\Rightarrow$??)

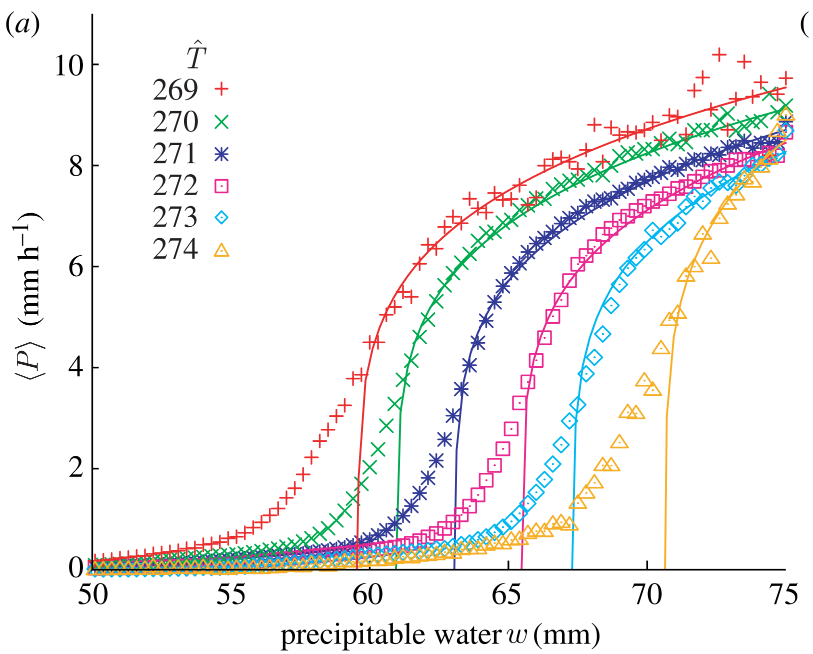
Think in the critical temperature of Fe, $T_c = 770°$C
or in the critical point of water, at $T_c = 374°$C and 218 atm

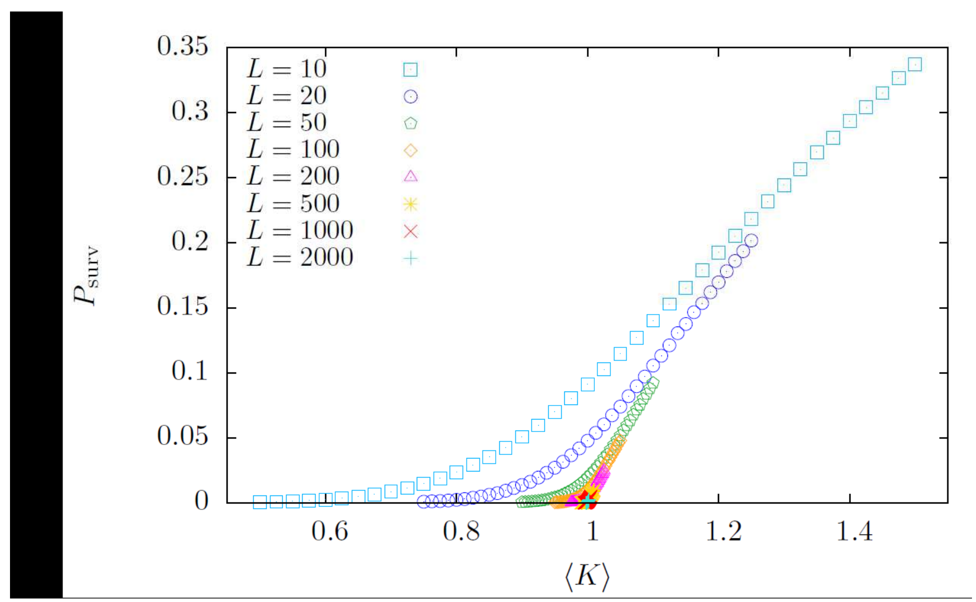
Andrews 1869 (for CO_2)

Other candidates for SOC

- For rain, Peters and Neelin have shown: *Nature Phys* 2006, Neelin et al. *Phil Trans R Soc A* 2008

  1. Existence in the atmosphere of a non-equilibrium stability-instability transition





Stanley, *Rev Mod Phys* 1999

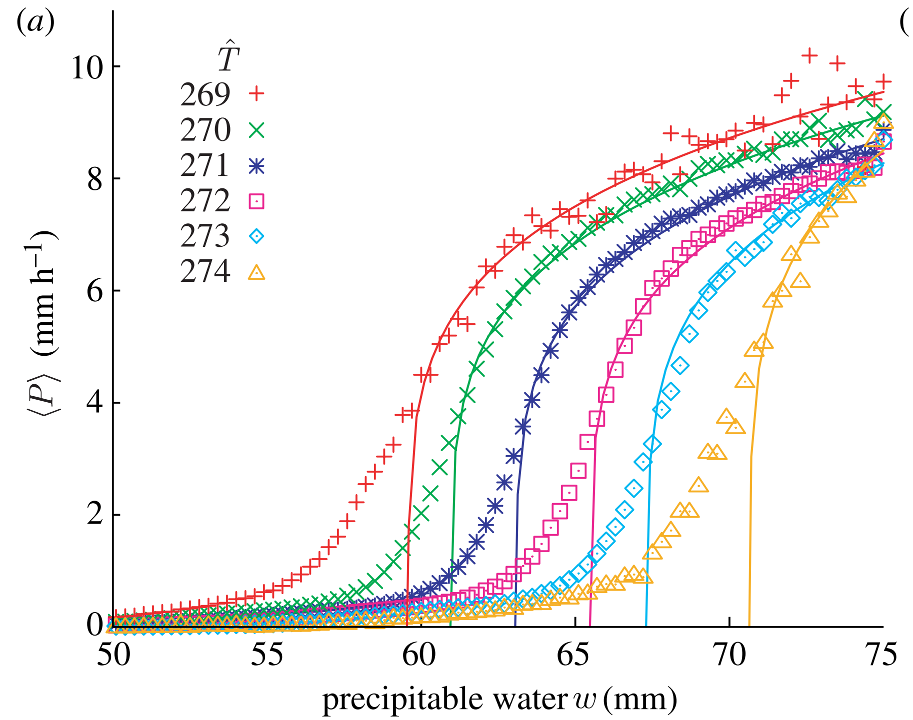
- Finite size effects

Finite size scaling: $\langle P\rangle = L^{-0.2/\nu} H[(w-w_c)L^{1/\nu}]$ (L system size)

$$H(x) \propto \begin{cases} |x|^{-\gamma} & \text{for } x\to -\infty \\ x^{0.2} & \text{for } x\to +\infty \end{cases} \underset{L\to\infty}{\Rightarrow} \langle P\rangle \propto \begin{cases} 0 & \text{for } w<w_c \\ (w-w_c)^{0.2} & \text{for } w>w_c \end{cases}$$

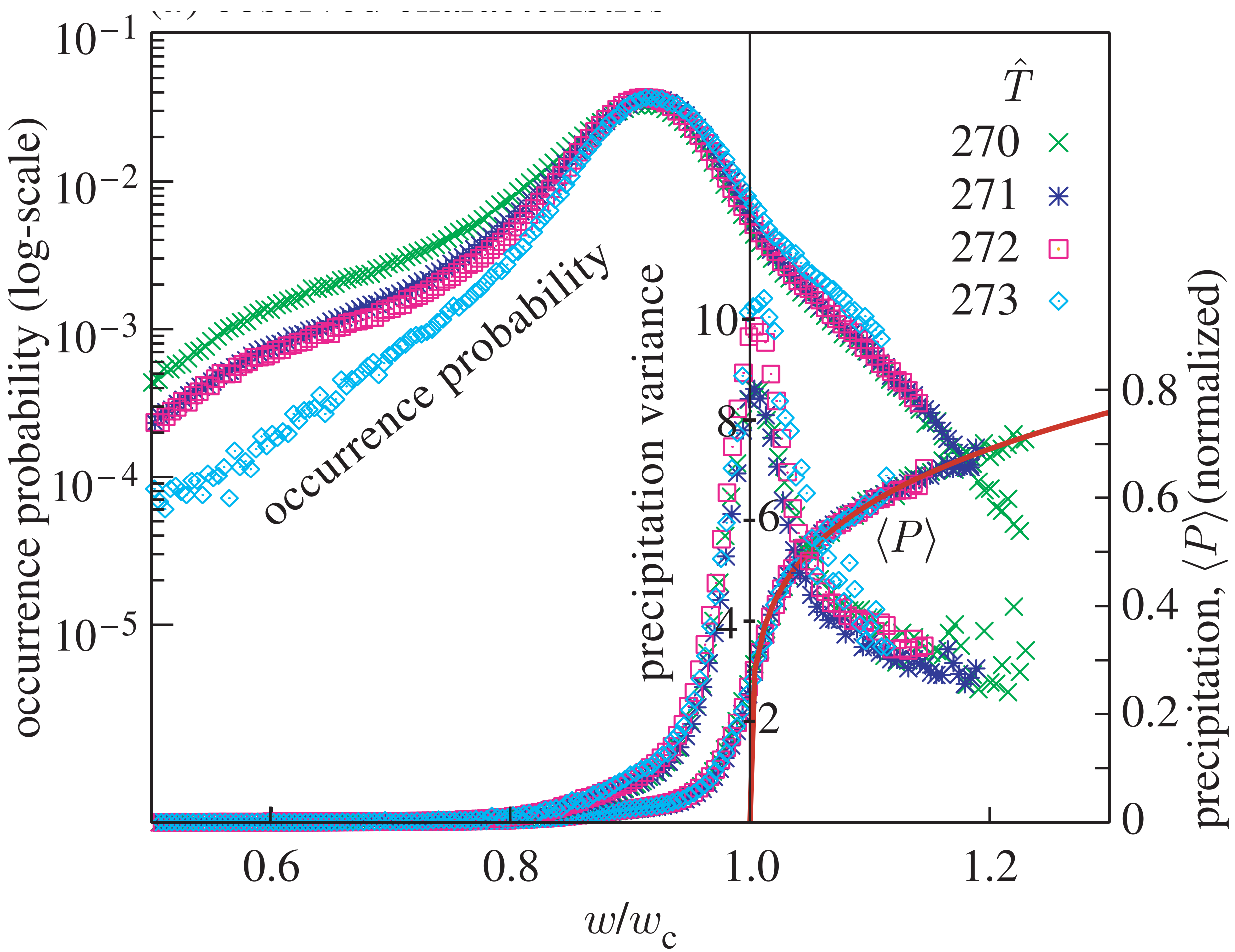
With critical point $w_c \simeq 63$ mm if $T = 271$ K, and so on

Phase transitions (abrupt changes) only exist in the limit $L\to\infty$



- Peters and Neelin have also shown: *Nature Phys* 2006, Neelin et al. *Phil Trans R Soc A* 2008

  2. The atmosphere is attracted towards the critical point of the transition



Continuous, non-upper-truncated power-law distributions

- Given by a probability density $D(x)$ with x real (continuous)

$$D(x) = \frac{B}{x^\gamma} \qquad \text{for } a \leq x < \infty$$

  with $a > 0$. Then, $\int_a^\infty D(x)dx = 1$ with $\gamma > 1$ implies

$$B = (\gamma - 1)a^{\gamma - 1}$$

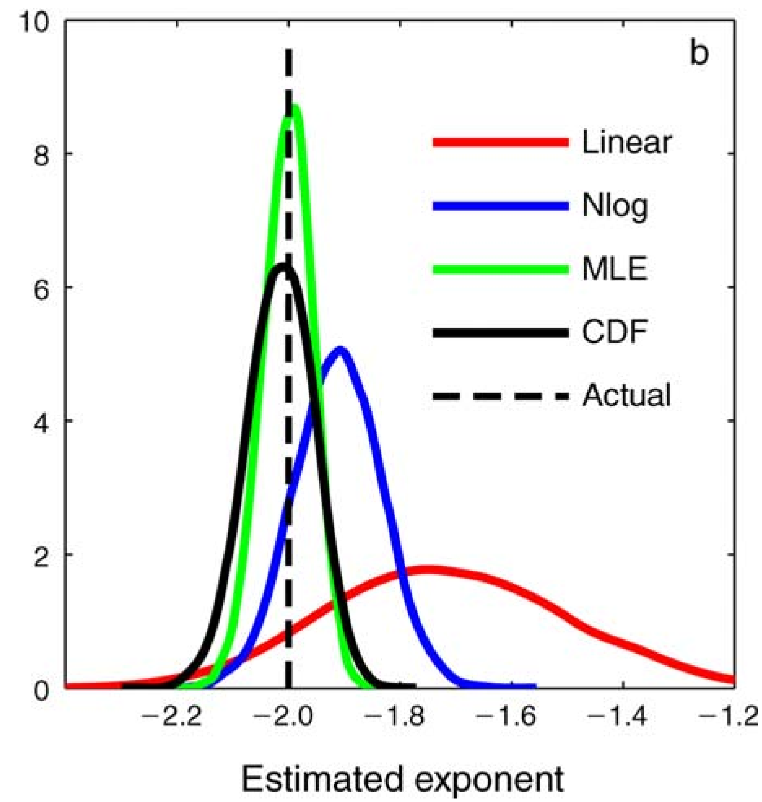
- In order to decide between competing explanations, universality classes, etc., it is important not only to determine if power laws hold, but also the precise value of the exponent γ

Fitting power-law distributions

- Most common method to fit power laws has been linear regression in log-log (needs to estimate first the empirical density $\Rightarrow$ delicate)

- Some authors have pointed out the superiority of maximum-likelihood (ML) estimation

  Goldstein et al. *Eur Phys J B* 2004
  Bauke, *Eur Phys J B* 2007
  White et al. *Ecol* 2008
  Clauset et al. *SIAM Rev* 2009



- ML estimators are: asymptotically unbiased and with lowest variance

  Invariant under re-parameterizations

Maximum likelihood (ML) estimation

- Given a dataset of size N, $x_1, x_2, \dots x_N$, the likelihood is the joint distribution

$$\mathcal{L}(\gamma) = D(x_1, x_2, \dots x_N; \gamma) = \prod_{i=1}^{N} D(x_i; \gamma)$$

(assuming independence). For a power law, $D(x) = B/x^{\gamma}$, the log-likelihood is

$$\ell(\gamma) = \frac{\ln \mathcal{L}(\gamma)}{N} = \frac{1}{N} \sum_{i=1}^{N} \ln D(x_i; \gamma) = \ln B - \frac{\gamma}{N} \sum_{i=1}^{N} \ln x_i = \ln B - \gamma \ln G$$

with G the geometric mean of the data and $B = (\gamma - 1)a^{\gamma - 1}$ (continuous case).
Maximizing

$$\frac{d\ell}{d\gamma} = 0 \qquad \Rightarrow \qquad \gamma = 1 + \frac{1}{\ln \frac{G}{a}}$$

- Note that the ML solution depends only on the geometric mean G

$$\gamma = 1 + \left(\ln \frac{G}{a}\right)^{-1}$$

So, any data, from any distribution, with the same G yields the same γ

Then, maximum likelihood should be called minimum unlikelihood

$\Rightarrow$ A goodness-of-fit test is necessary

Goodness-of-fit test

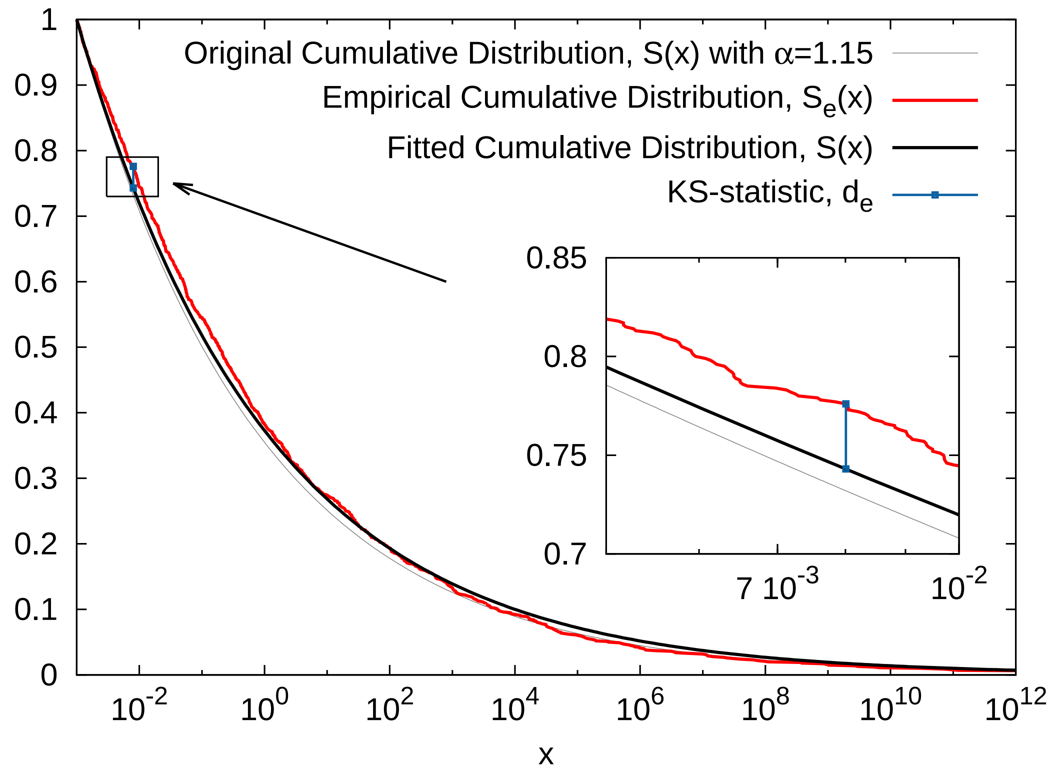
- In order to test the goodness of the (ML) fit let us consider Kolmogorov-Smirnov

  KS distance or KS statistic $=$ maximum difference for all x between empirical $S(x)$ and theoretical $S(x)$

  $d =$ max$\{S_{emp}(x) - S_{th}(x)\}$

  with $S(x) = \int_x^\infty D(x)dx$

  Care with $p-$value: Monte Carlo simulations



- The problem of power-law fitting is not in fitting the power law

  It is in the selection of $a =$ minimum value of x

Clauset's et al. recipe

Clauset, Shalizi & Newman *SIAM Rev* 2009

- ⋆ Take an arbitrary value of a ($=$ minimum x for which the power law holds)
  ⋆ Calculate fit by ML estimation $\Rightarrow$ yields exponent γ
  ⋆ Calculate Kolmogorov-Smirnov distance between empirical distribution and fit (no goodness-of-fit yet)

- ⋆ Select the value of a which minimizes Kolmogorov-Smirnov distance $d = d_{emp}$
  So, we come out with a fit given by a_{emp} and γ_{emp}

- ⋆ Calculate $p-$value of the fit by simulating N_{sim} synthetic samples:
  - $*$ Simulating a power law with exponent γ_{emp} for $x \geq a_{emp}$
  - $*$ Bootstrap of the empirical distribution for $x < a_{emp}$

  ⋆ Proceed with synthetic samples in exactly the same way as with the empirical

$\Rightarrow$ Each synthetic sample yields a value of d_{sim}

$\Rightarrow$ Calculate $p-$value as $\quad p \simeq \{\text{number of } d_{sim} > d_{emp}\}/N_{sim}$

- Justification of the minimization of d

  Under the null hypothesis, Kolmogorov-Smirnov distance goes as

  $$d \propto \frac{1}{\sqrt{N}}$$

  So, under the null hypothesis, the smaller a, the larger N and the smaller d

  But as soon as the null hypothesis fails, the fit deviates and d increases

  A sort of balance between the two effects is implicit

  Nevertheless, there is no reason why this deviation should compensate and overcome the reduction in d

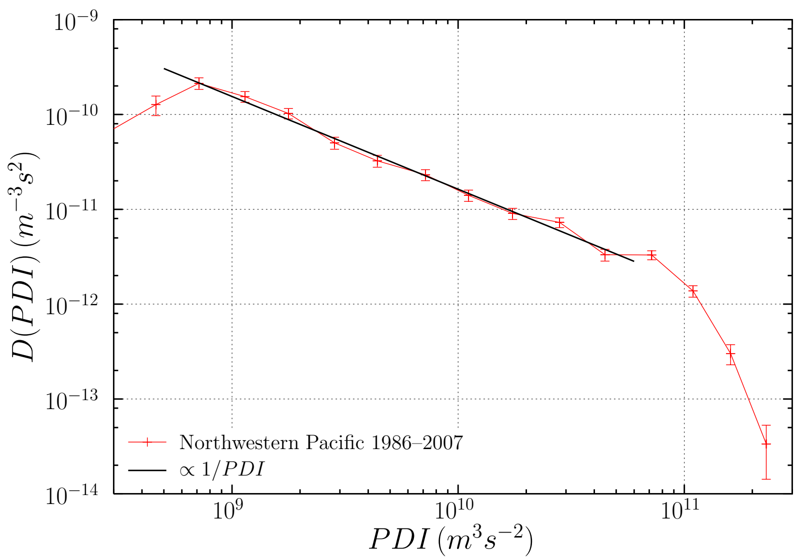
  (it would depend on the shape of the distribution for $x < a$)

Problems of Clauset et al.'s recipe

- The method performs bad when generalized to truncated power laws

$$D(x) \propto \frac{1}{x^\gamma} \qquad \text{for } a \le x \le b$$

This is common in complex systems, due to finite-size effects

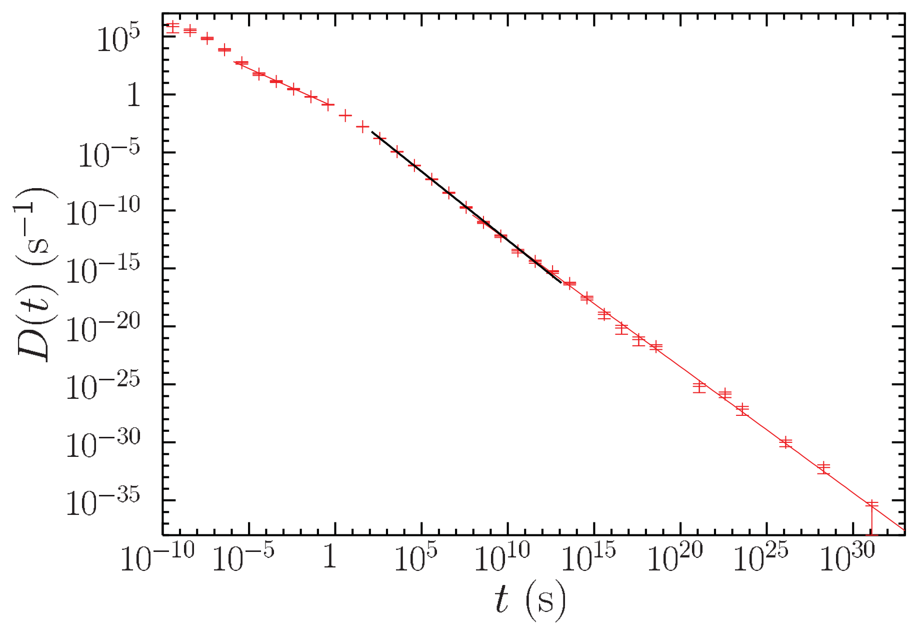


More problems of Clauset et al.'s recipe

A.C., Font & Camacho *Phys Rev E* 2011

- Consider nuclear half-lives: from below 10^{-16} s to 10^{23} yr $\sim$ 10^{31} s for ^{128}Te

  Clauset et al.'s recipe yields
  $a_{emp} = 30$ s and $\gamma_{emp} = 1.16$
  but $p = 0$

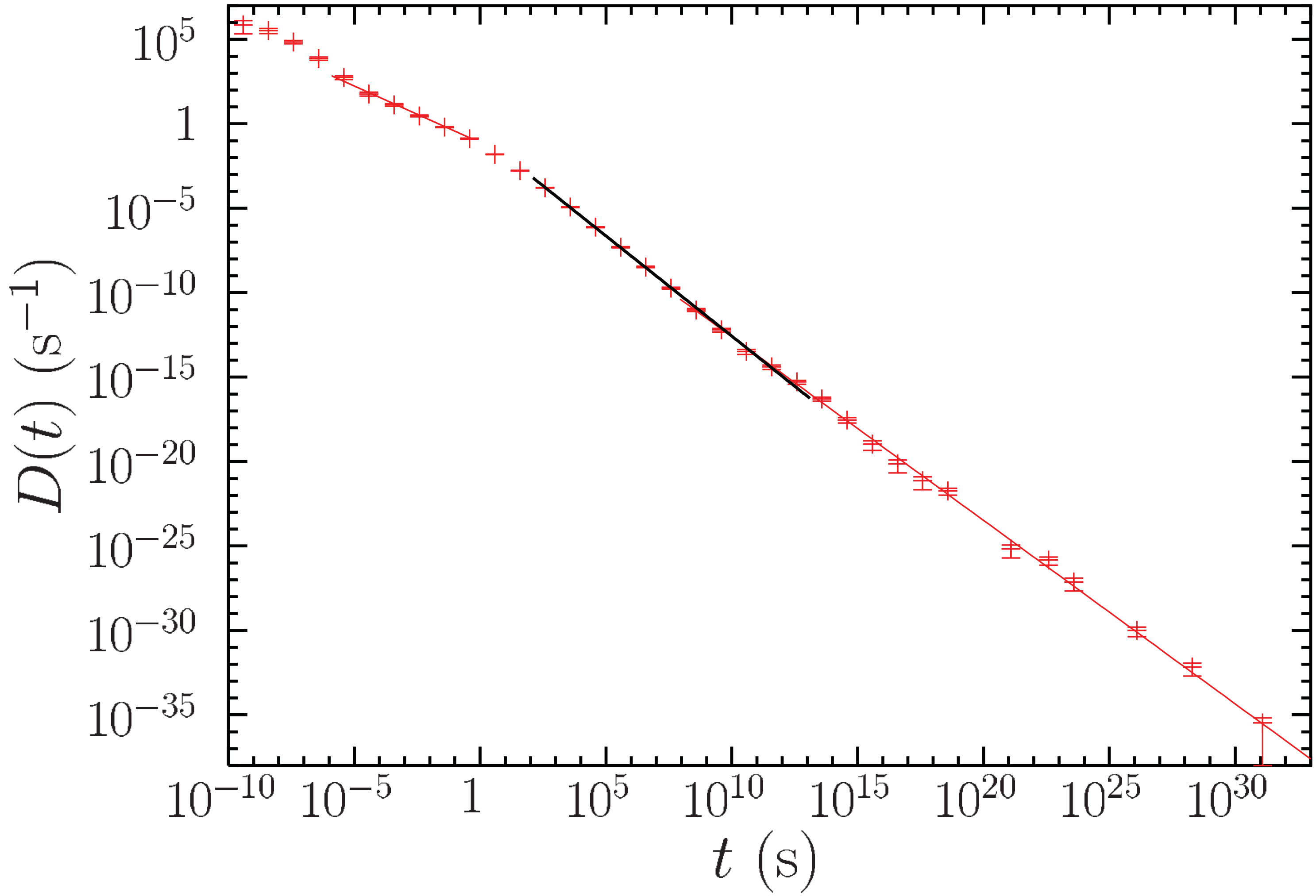


- Simulate power law for $x \geq 10^8$ s and bootstrap original data for $x < 10^8$ s

  In 80 % of the cases $p = 0$

  So, the recipe leads to the usual rejection of the power-law hypothesis when it is true!

  (p should be uniformly distributed between 0 and 1 under the null hypothesis)



⇒ Failure of the Clauset's et al. recipe

Alternative recipe

Peters et al. *J Stat Mech* 2010; Deluca & A.C. *Acta Geophys* 2013

- ⋆ Take an arbitrary value of a ($=$ minimum x for which the power law holds)
  ⋆ Calculate fit by ML estimation $\Rightarrow$ yields exponent γ
  ⋆ Calculate KS distance d between empirical distribution and fit
  (no difference with Clauset et al. yet)

- ⋆ Calculate a $p-$value for fixed a
  ∗ Simulate N_{sim} power-law synthetic samples with γ for $x \geq a$
  ⋆ Proceed with synthetic samples in exactly the same way as with the empirical

  $\Rightarrow$ Each synthetic sample yields a value of d_{sim}

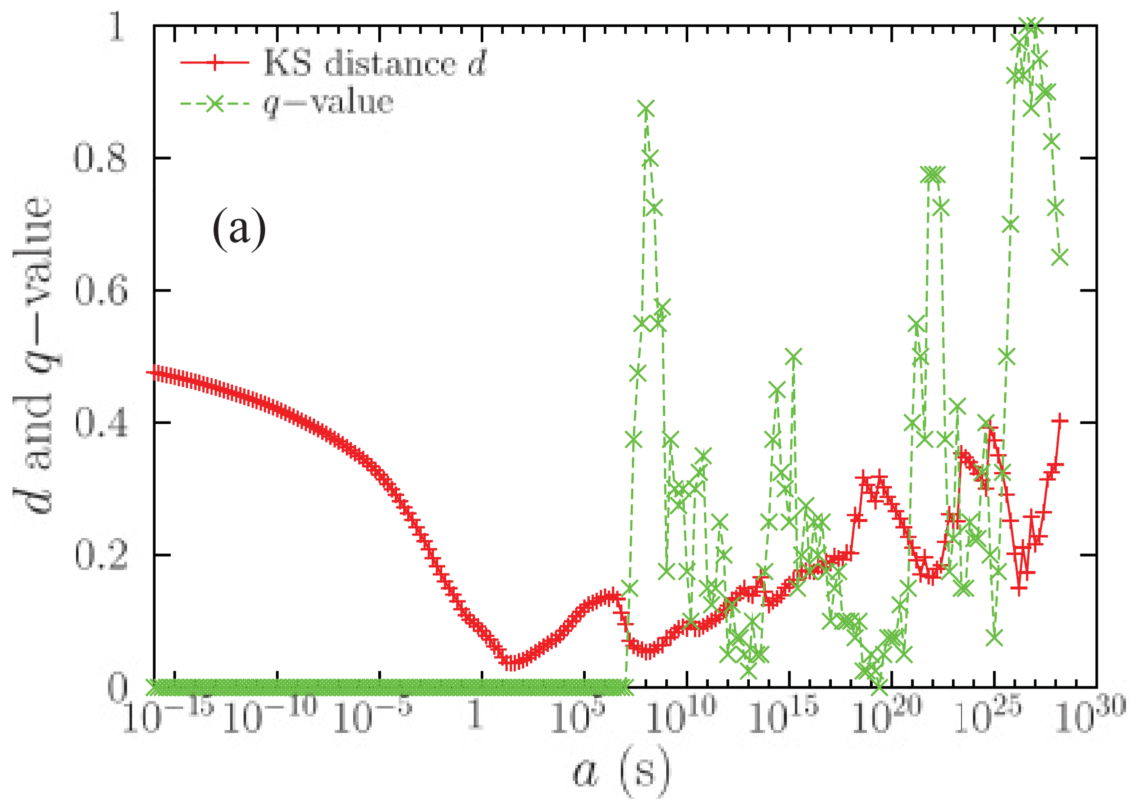
  ⋆ Calculate p as

$$p \simeq \frac{\text{number of } d_{sim} > d}{N_{sim}}$$

- Select the smallest value of a provided that $p > 0.20$ (e.g.)

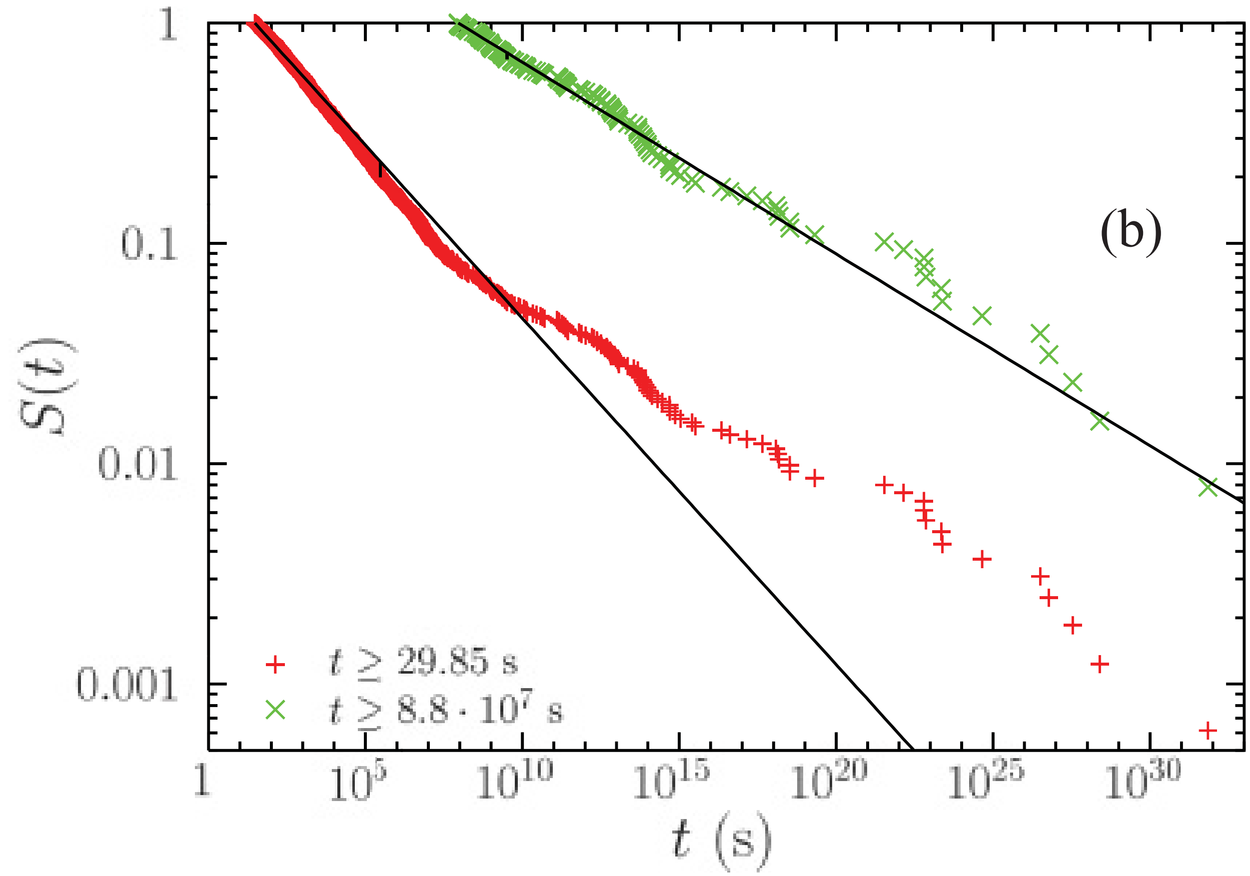
Performance

- Consider again nuclear half-lives: d and $p(=q)$ versus a

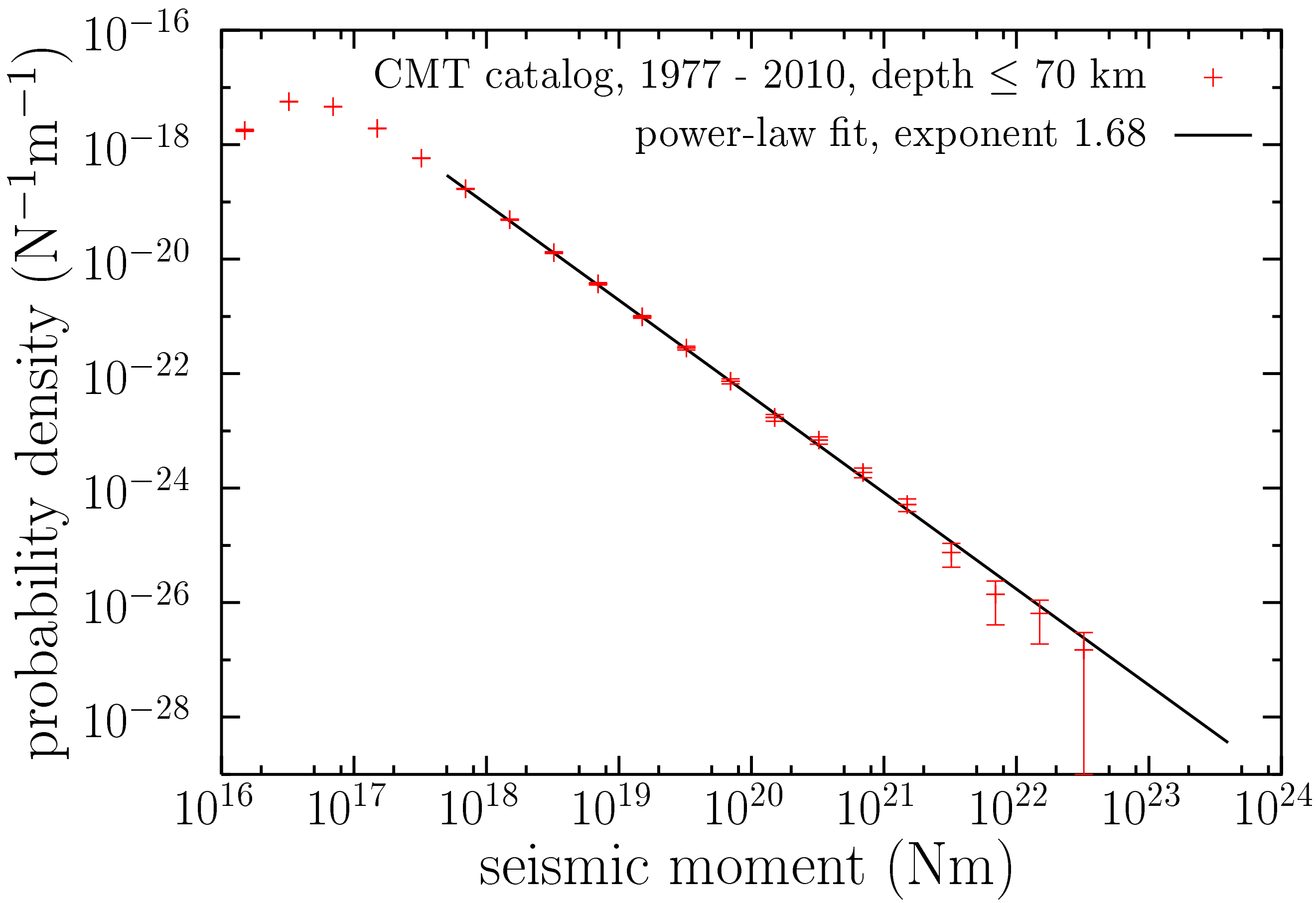


We obtain $a_{emp} = 3 \times 10^7$ s (~ 1 yr) and $\gamma_{emp} = 1.09$ (with $p > 0.20$)

- Comparison between Clauset et al.'s solution (red) (power-law rejected) and alternative (green) (for the (complementary) cumulative distribution)



- Global earthquakes revisited: power law cannot be rejected



But other fits are possible! $\Rightarrow$ LRT, or AIC, or BIC... Main et al. *Nature Geosci* 2008

(Upper) Truncated power laws

- Deviations from a power law arise for large x, due to finite size effects. So,

$$D(x) = \frac{B}{x^\gamma} \quad \text{with } a \le x \le b \quad \text{and } B = \frac{(\gamma-1)a^{\gamma-1}}{1-(a/b)^{\gamma-1}}$$

Be careful: b is not $b-$value. The log-likelihood is now

$$\ell(\gamma) = \ln B - \gamma \ln G = -\gamma \ln \frac{G}{a} - \ln a + \ln \frac{\gamma-1}{1-(a/b)^{\gamma-1}}$$

with G the geometric mean of data between a and b.

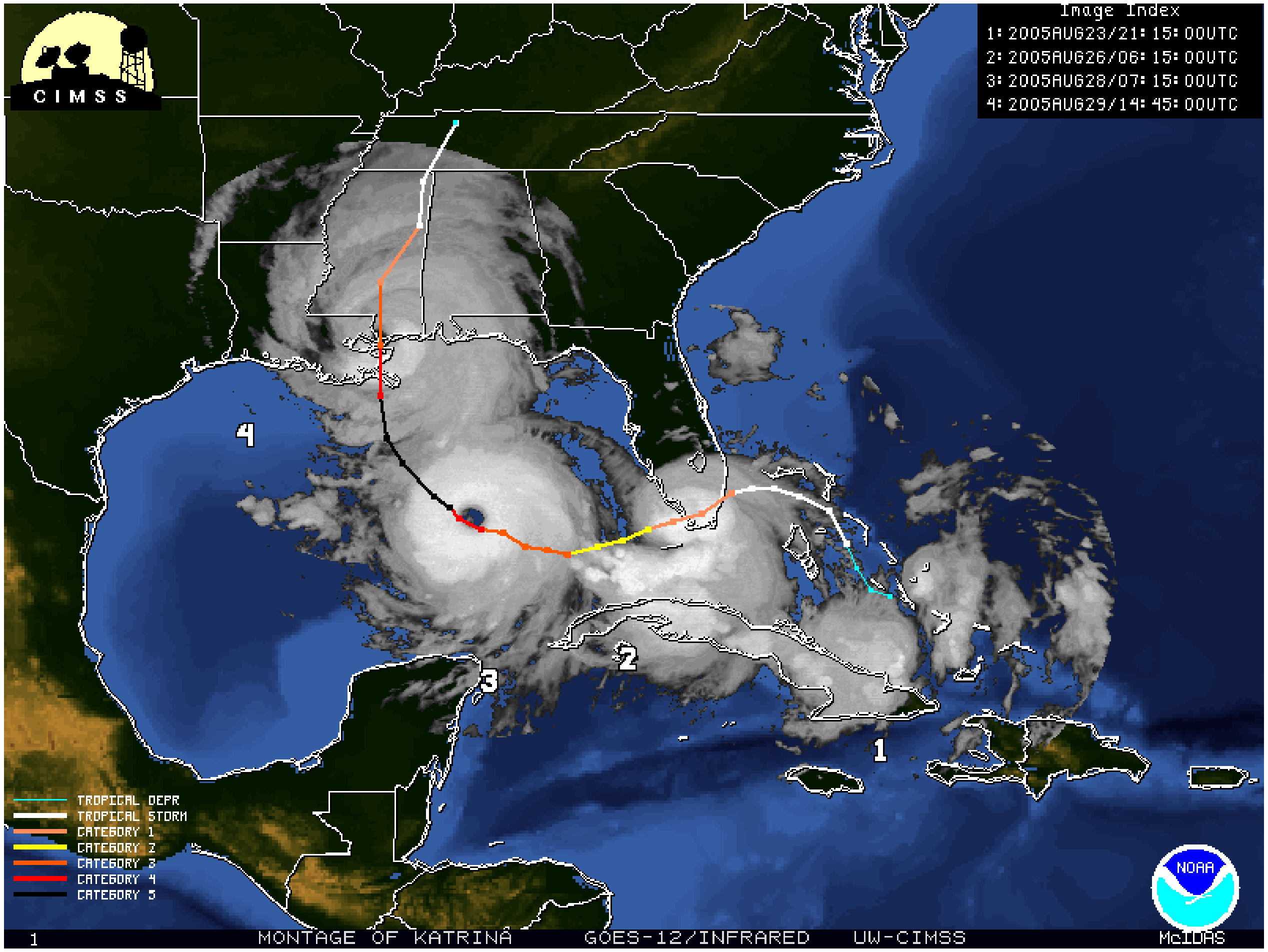
The log-likelihood needs to be maximized numerically

But the rest of the method is the same, swapping both a and b

Tropical cyclones

- (hurricanes, typhoons)

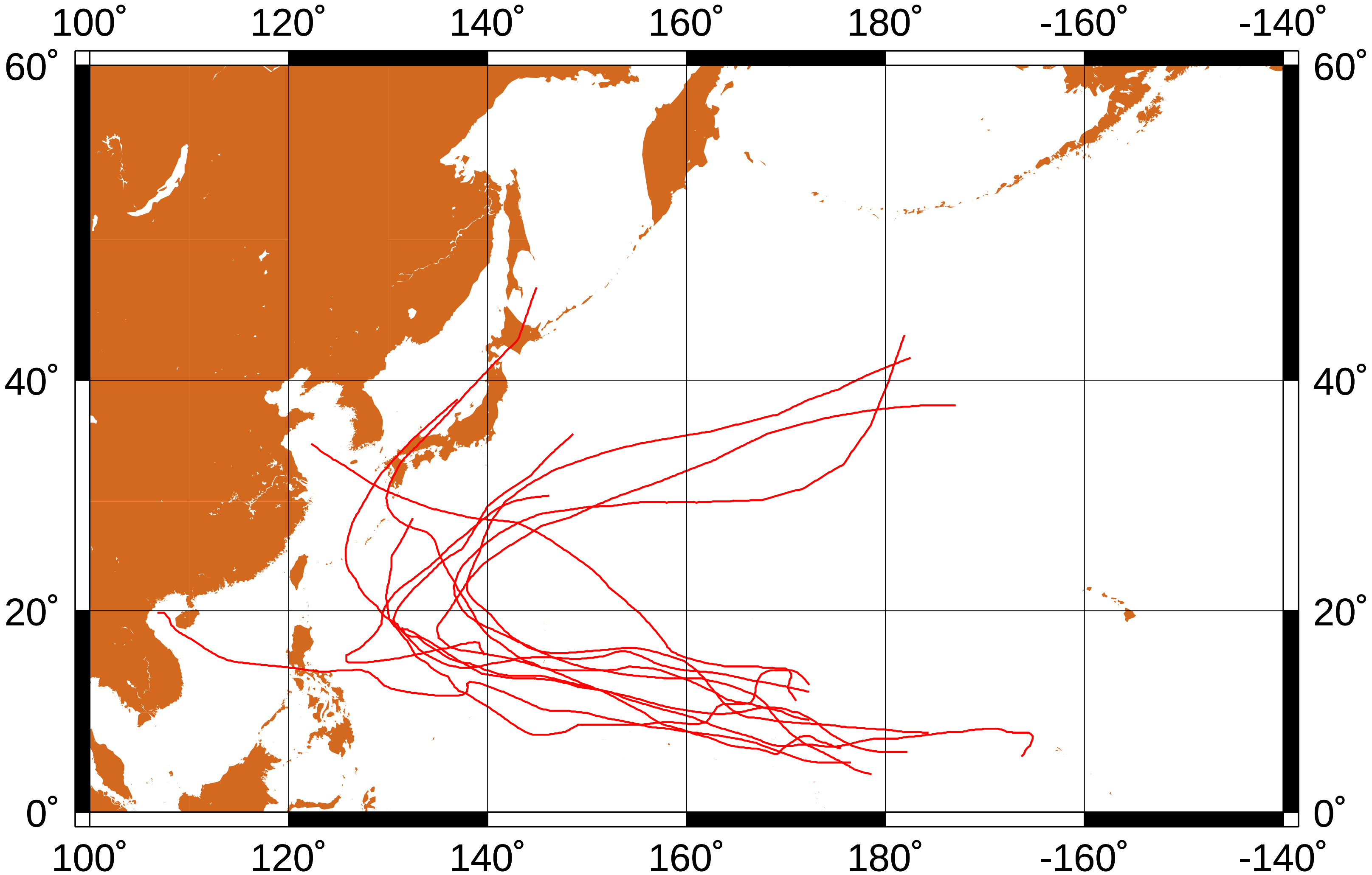
http://cimss.ssec.wisc.edu



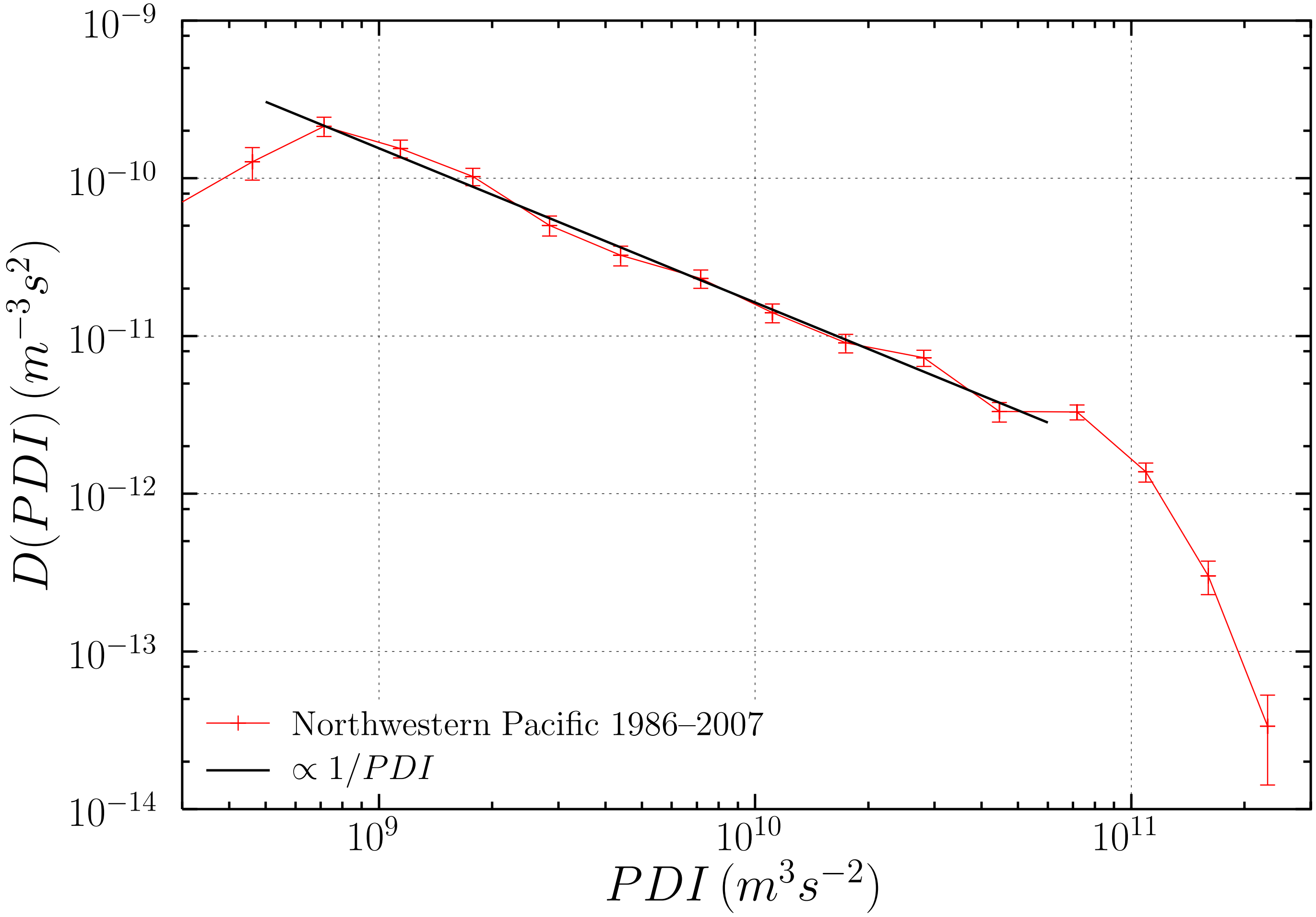
$$\text{Energy} \simeq \int\int C_D \rho |v(r,t)|^3 d^2 r dt$$

Bister & Emanuel, *Met Atm Phys* 1998

- Typhoons in the North Western Pacific (only the largest ones)

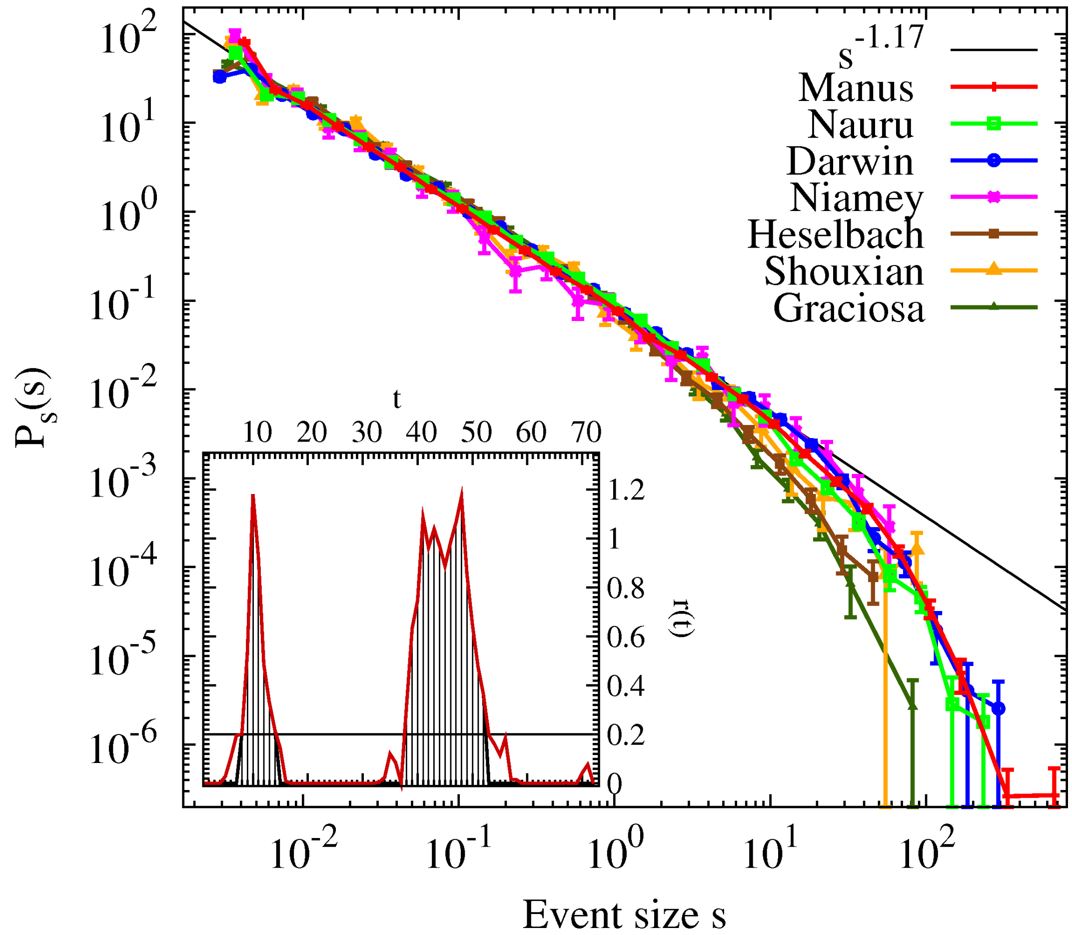


- Tropical cyclones (hurricanes, typhoons), A.C., Ossó, Llebot, *Nature Phys* 2010



- Rainfall

Peters et al. *Stat Mech* 2010



Discrete power laws

A. C., A. Deluca, & R. Ferrer-i-Cancho, *ArXiv* 2012

- The probability function is given by

$$f(n) = \frac{B}{x^\gamma} \quad \text{with } x = a, a+1, \dots, \text{ and } \gamma > 1 \quad \text{and } B = \frac{1}{\zeta(\gamma, a)}$$

where $\zeta(\gamma, a)$ is the Hurwitz theta function ($\zeta(\gamma, 1)$ is the Riemann function)

The log-likelihood is

$$\ell(\gamma) = -\ln \zeta(\gamma, a) - \gamma \ln G$$

which is more difficult to maximize

Care with the cumulative distribution function (for the KS test)

The simulation of the discrete distribution is more involving also

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- `http://www.crm.cat/en/About/People/Researchers/ACorral`