Theory of branched flows and the statistics of extreme waves in random media

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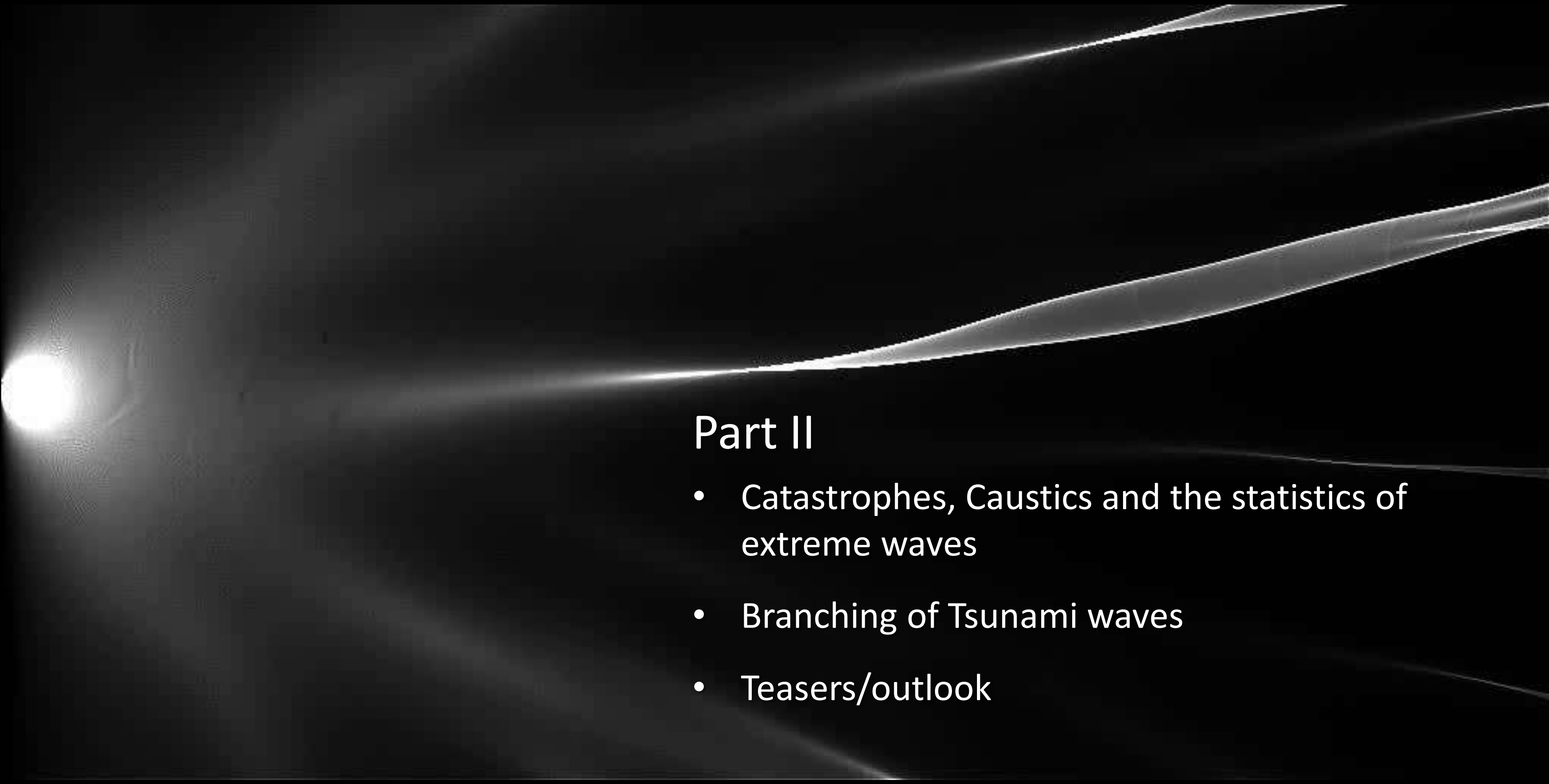

Part II

- Catastrophes, Caustics and the statistics of extreme waves
- Branching of Tsunami waves
- Teasers/outlook

Overview

Part I

- The phenomenon of branching
- In which systems can it be observed?
- Statistics of random caustics
- Characteristic transport length scale
- Wave intensity/height statistics

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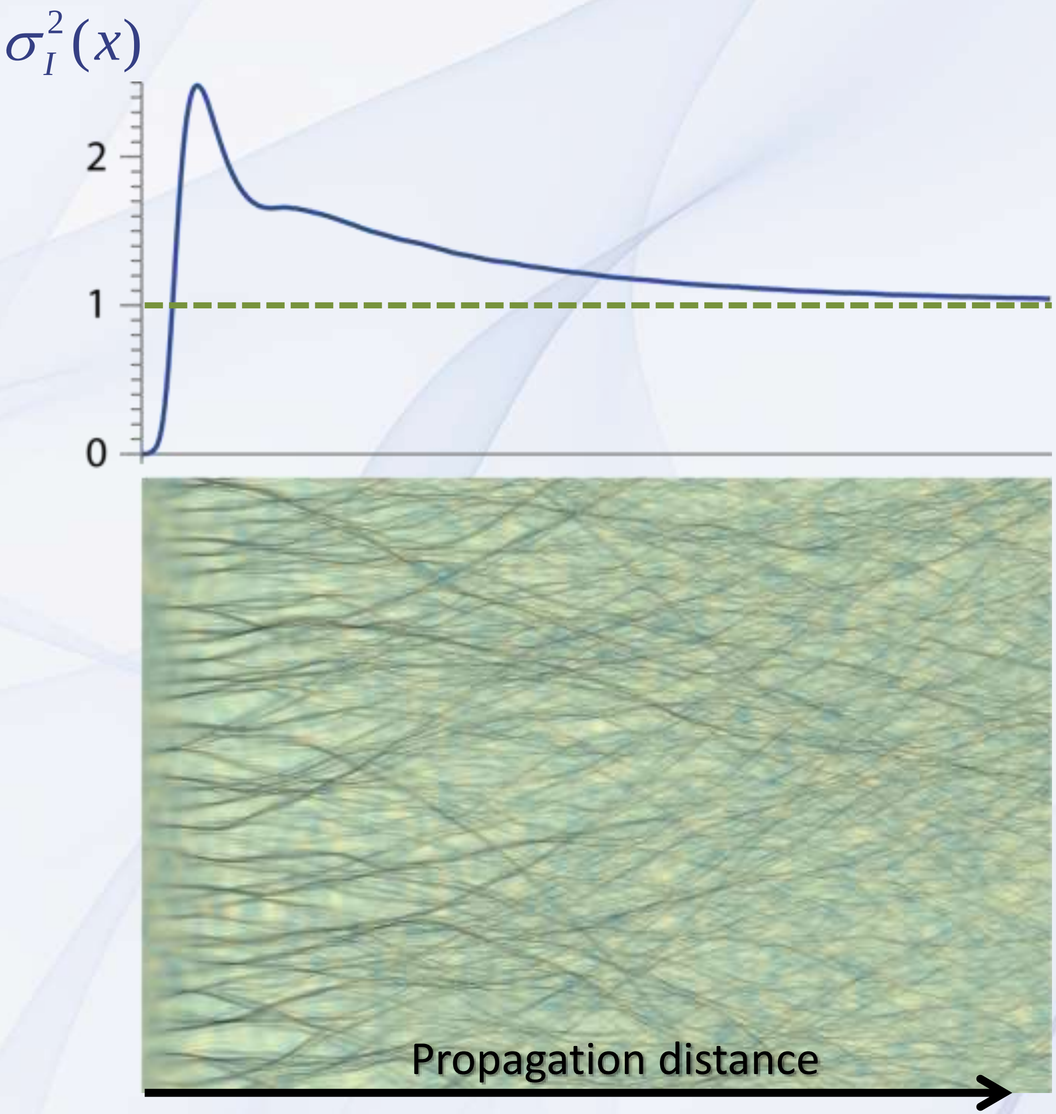
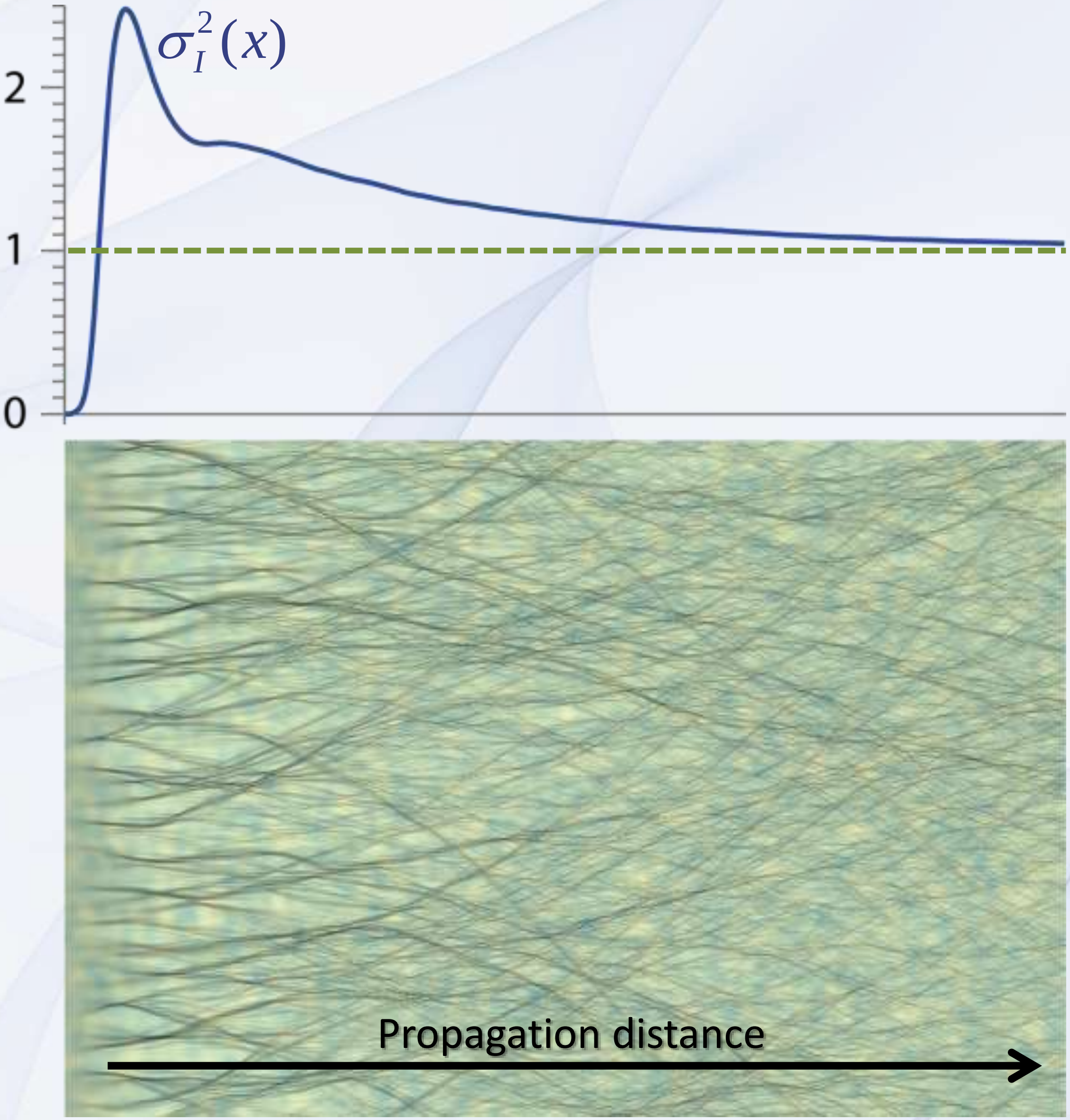
Scintillation index



$$\sigma_I^2(x) = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2}$$

$$= \frac{\text{variance of the intensity}}{(\text{mean intensity})^2}$$

Theory of intensity statistics



Predictions for the intensity distribution in branched flows:

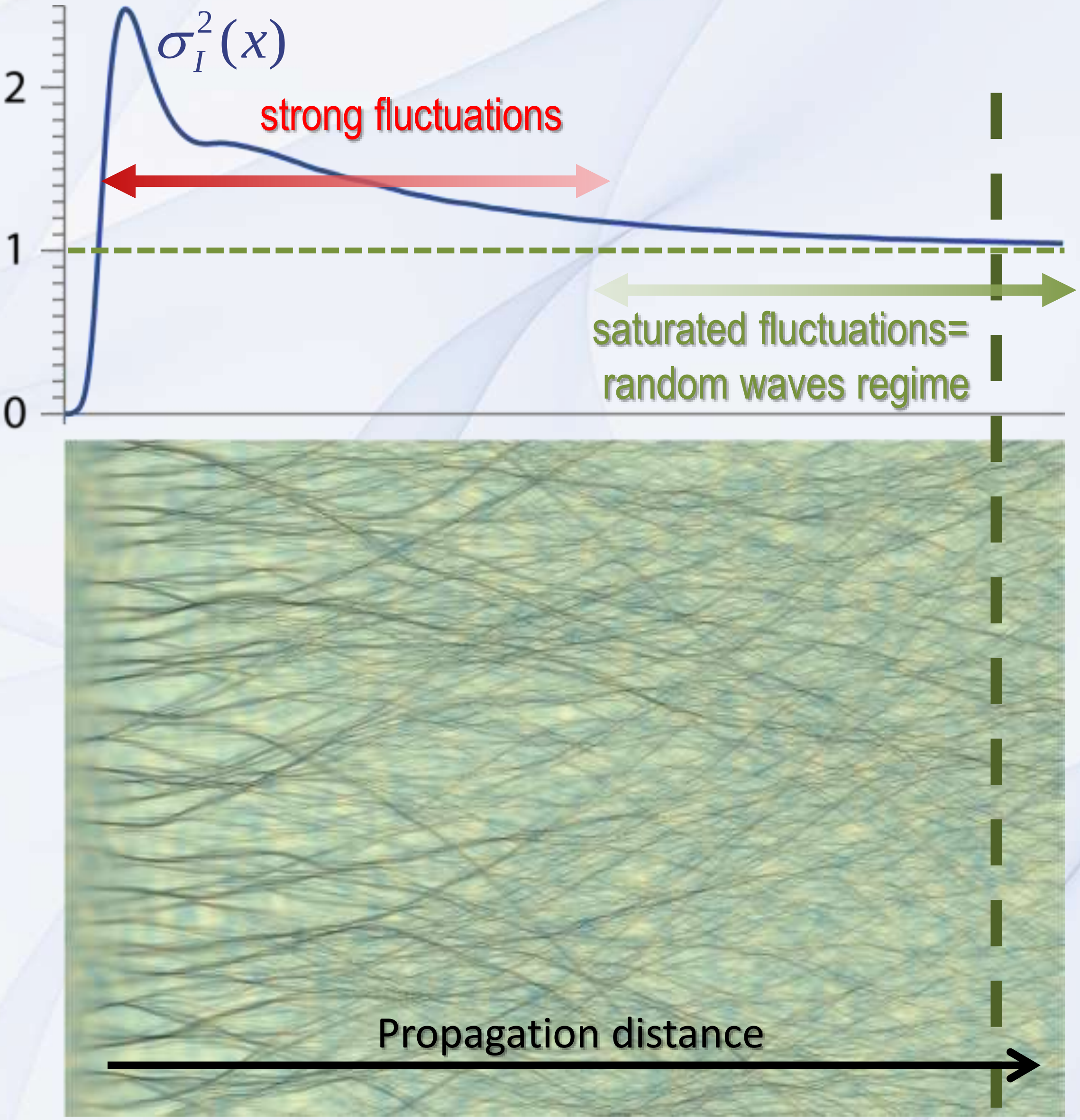
- Log-normal distribution

  Wolfson & Tomsovic JASA 2001

- Power-law tails

  Kaplan PRL 2002

Theory of intensity statistics



- Saturated regime reached long before the mean free path!

Random waves intensity

$$\psi = \sum_j \psi_j = \sum_j a_j + i \sum_j b_j$$

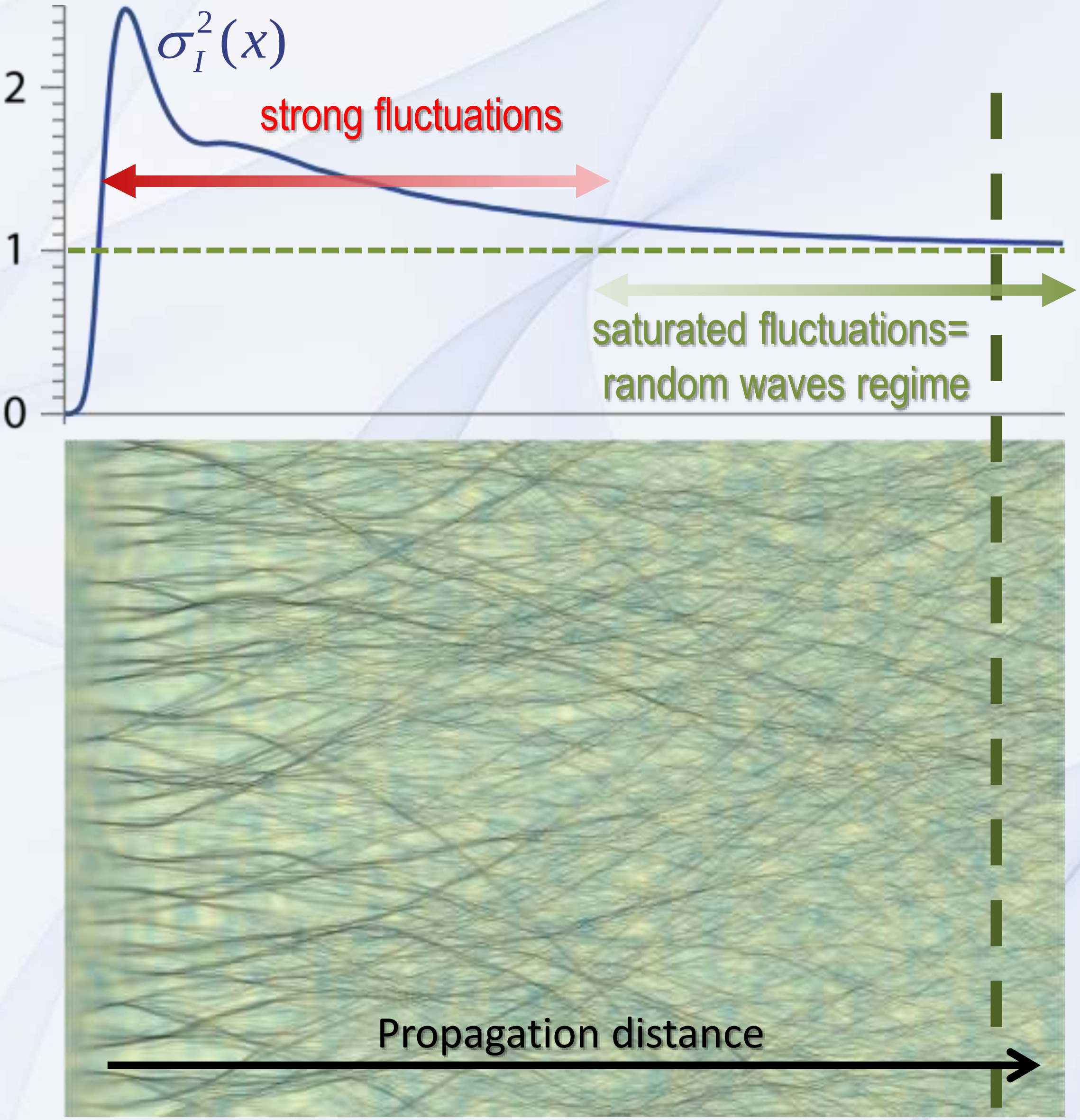
$$\Rightarrow I = |\psi|^2 = \left|\sum_j a_j\right|^2 + \left|\sum_j b_j\right|^2 = z_1{}^2 + z_2{}^2 \text{ with } z_1 \text{ and } z_2 \text{ Gaussian distributed}$$

$$P(I) = \frac{1}{\langle I \rangle} e^{-\frac{I}{\langle I \rangle}}$$

random wave intensities are exponentially distributed

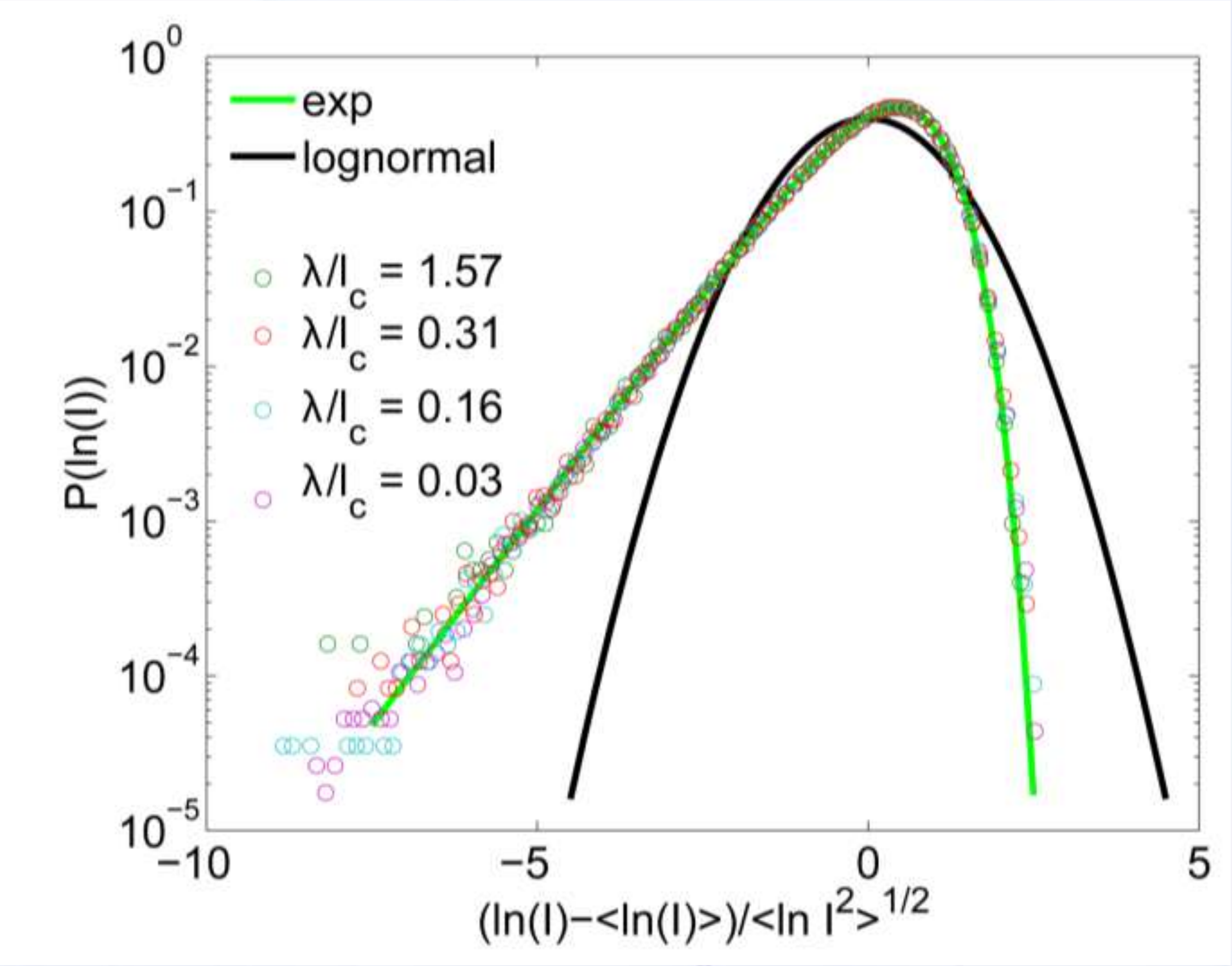
$$\left[P\left(\sqrt{I}\right) = \text{Rayleigh distributed} \right]$$

Theory of intensity statistics



- Saturated regime reached long before the mean free path!

Saturated fluctuations



Classical/ray-intensity

$$M(t,t_0)=\begin{pmatrix}\dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial v_0}\\[2ex] \dfrac{\partial v}{\partial y_0} & \dfrac{\partial v}{\partial v_0}\end{pmatrix}$$

$$M(t_n,t_0)=\prod_{i=0}^{n} M(t_i+1,t_i)$$

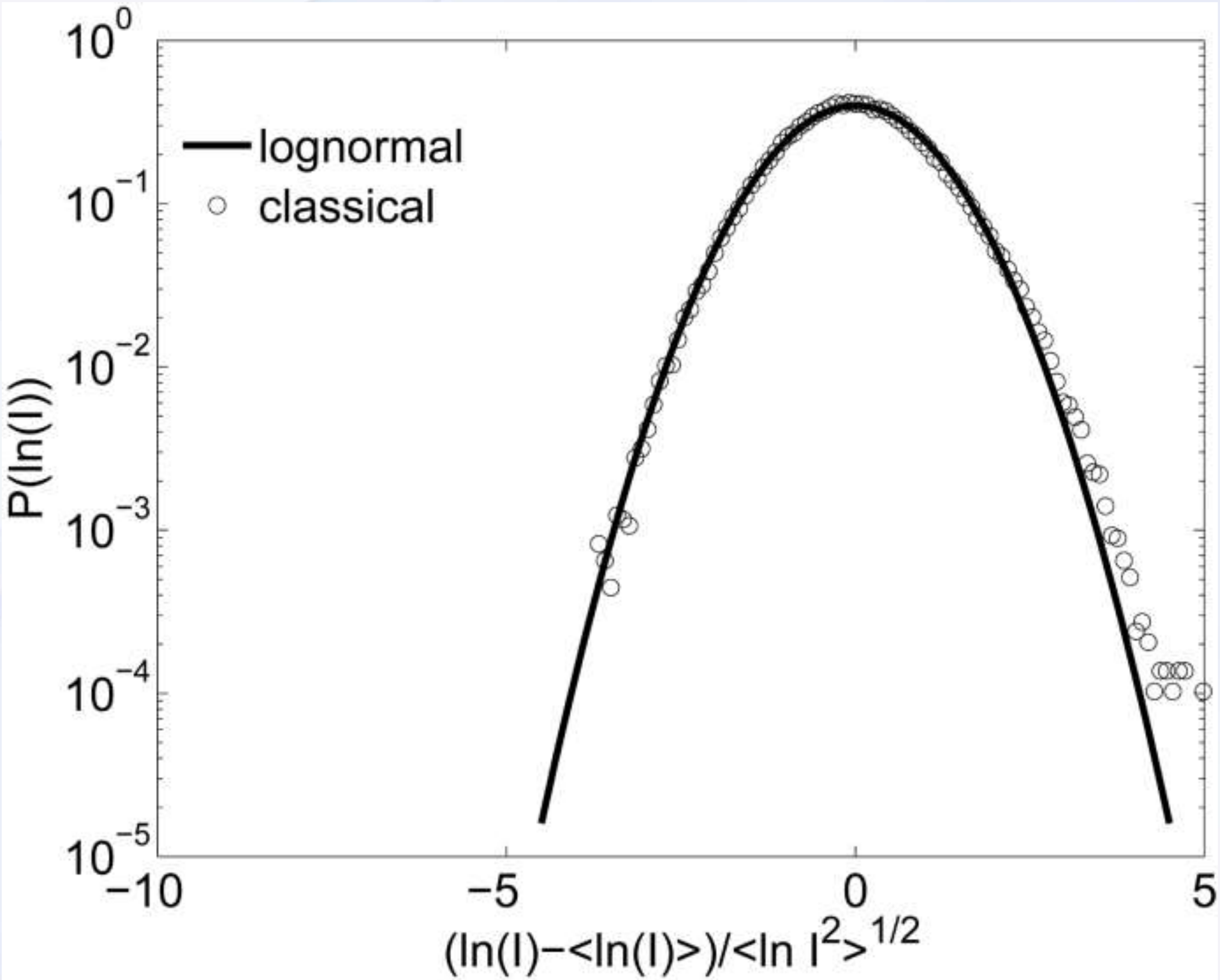
$$\Rightarrow \mathrm{Tr}\left(M(t,t_0)\right)\ \text{follows a log-normal-distribution}$$

$$\rho \propto \left|\frac{\partial y}{\partial y_0}\right|^{-1}$$

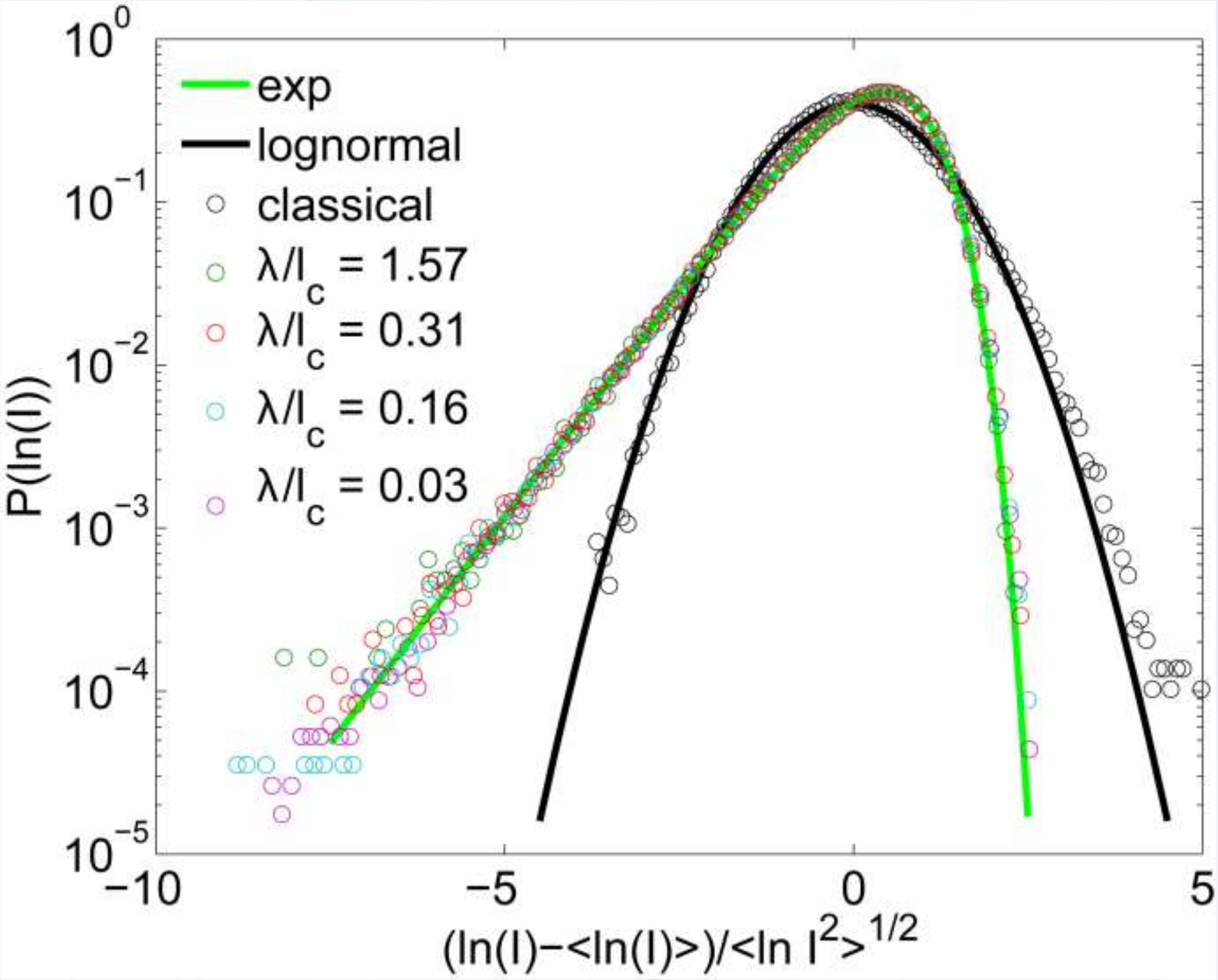
log-normally distributed?

Michael A. Wolfson and Steven Tomsovic, J. Acoust. Soc. Am. **109**, 2693 (2001).

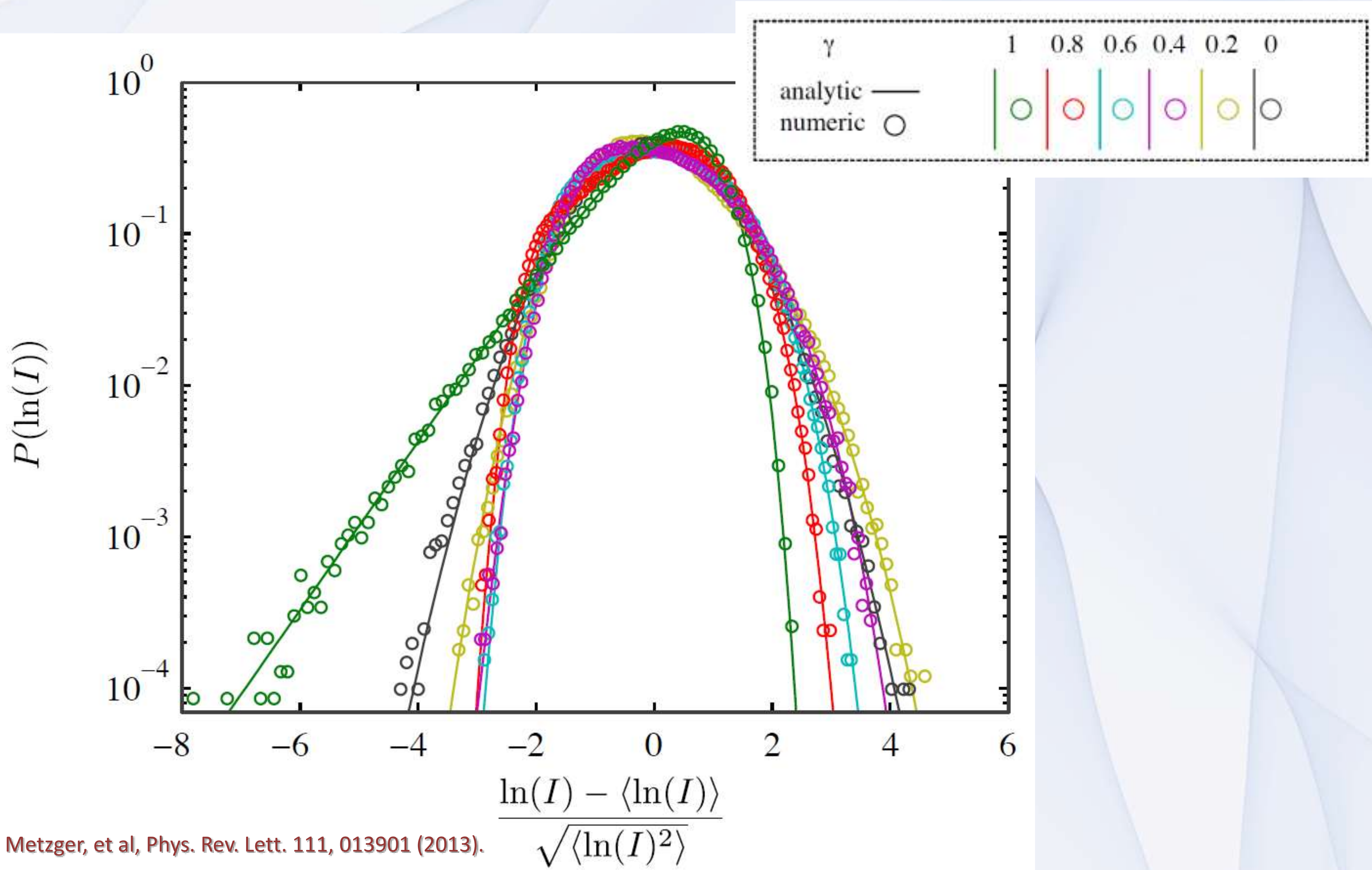
Classical/ray-intensity



Intensity statistics



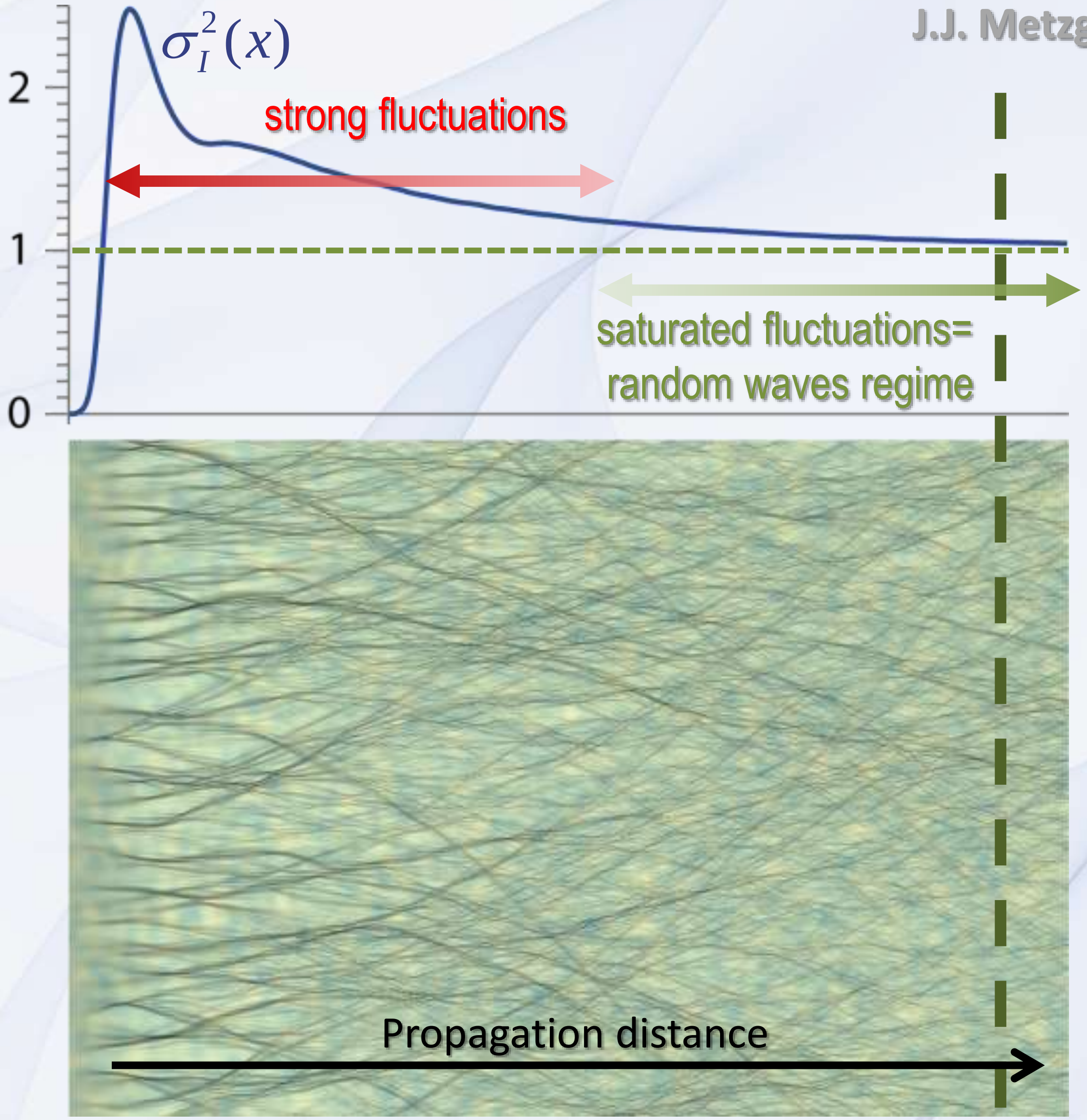
Intensity fluctuations governed by coherence



Metzger, et al, Phys. Rev. Lett. 111, 013901 (2013).

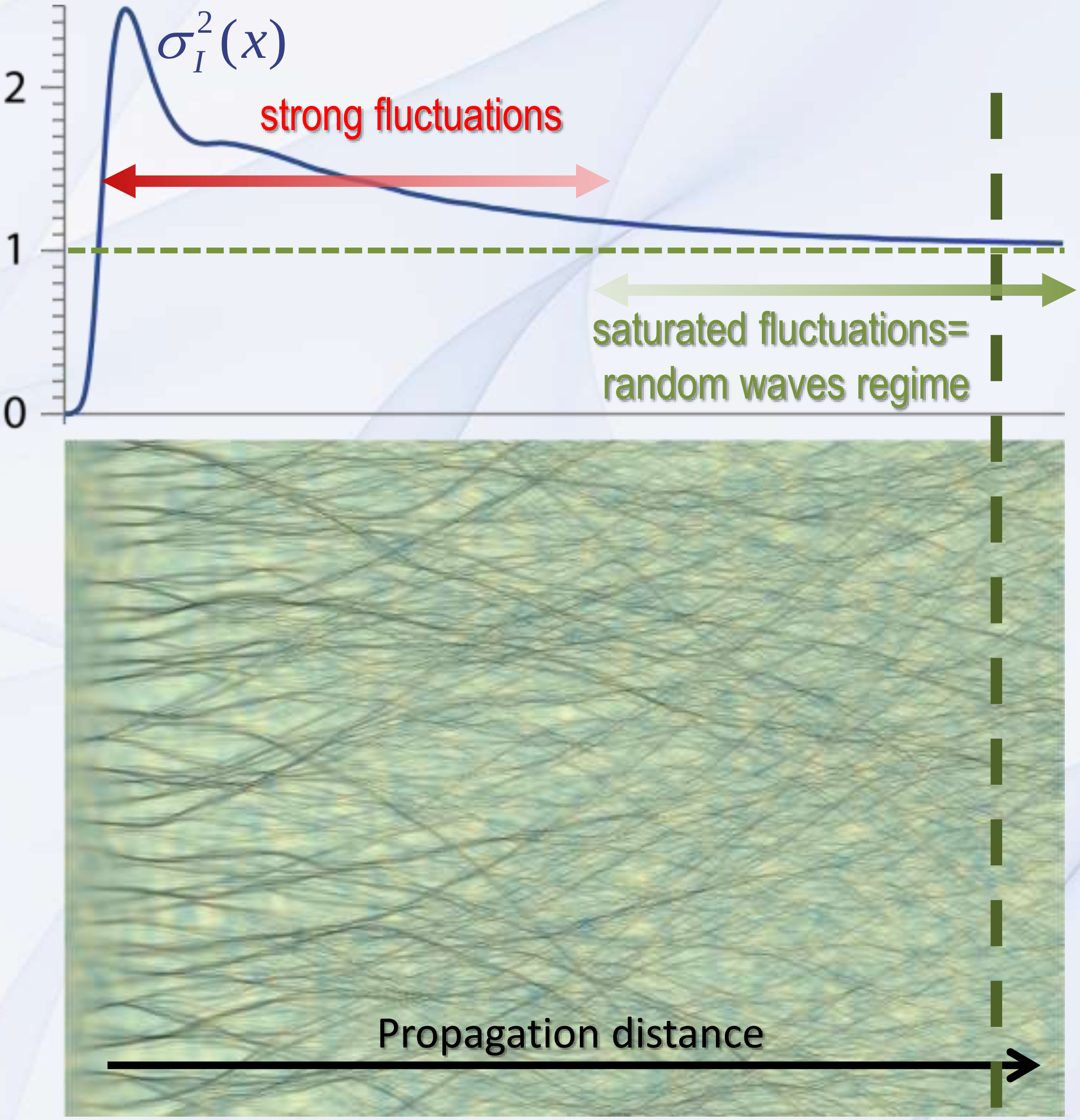
Theory of intensity statistics

J.J. Metzger, R. Fleischman, T. Geisel PRL 2013



- Saturated regime reached long before the mean free path!
- Coherent waves show Rayleigh distribution
- Decoherent waves show log-normal distribution
- Theory for all degrees of coherence

Theory of intensity statistics



Predictions for the intensity distribution in branched flows:

- Log-normal distribution

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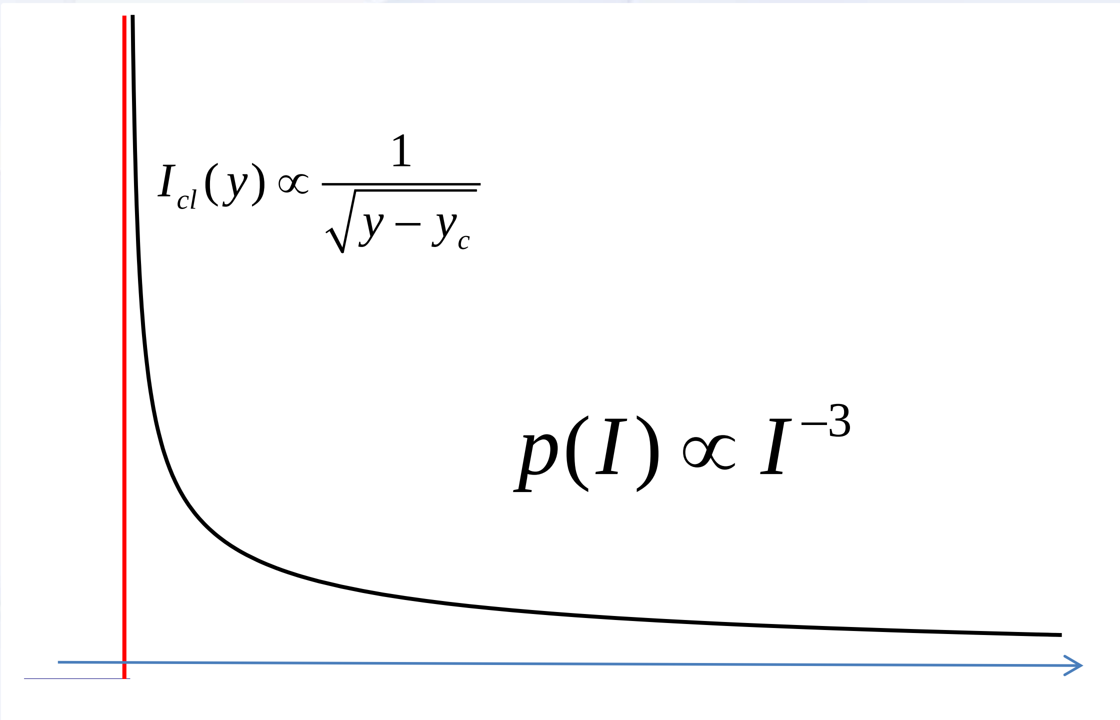
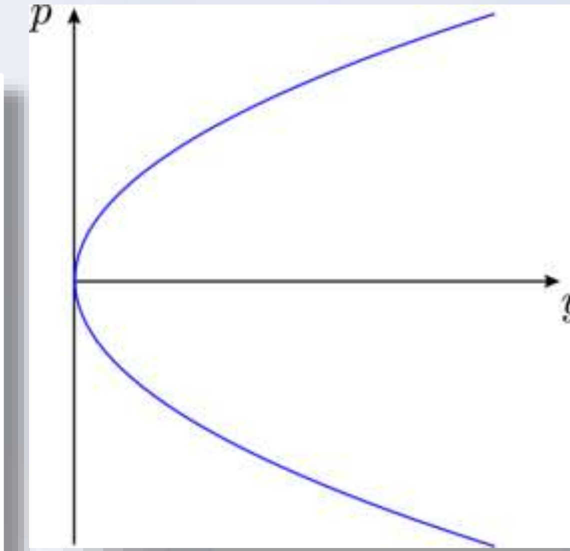
- Power-law tails

  Kaplan PRL 2002

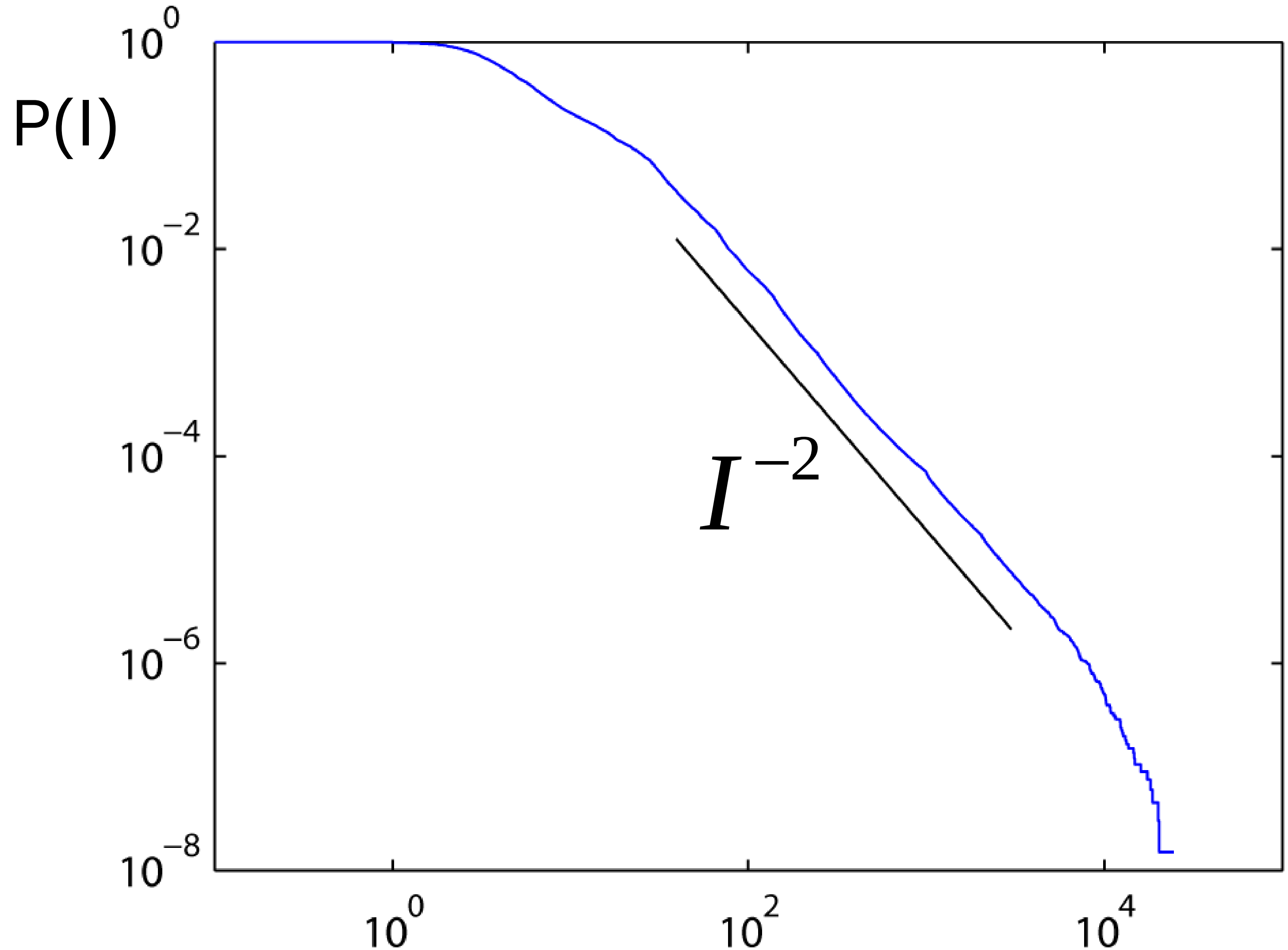
Ray-Intensity at a Caustic



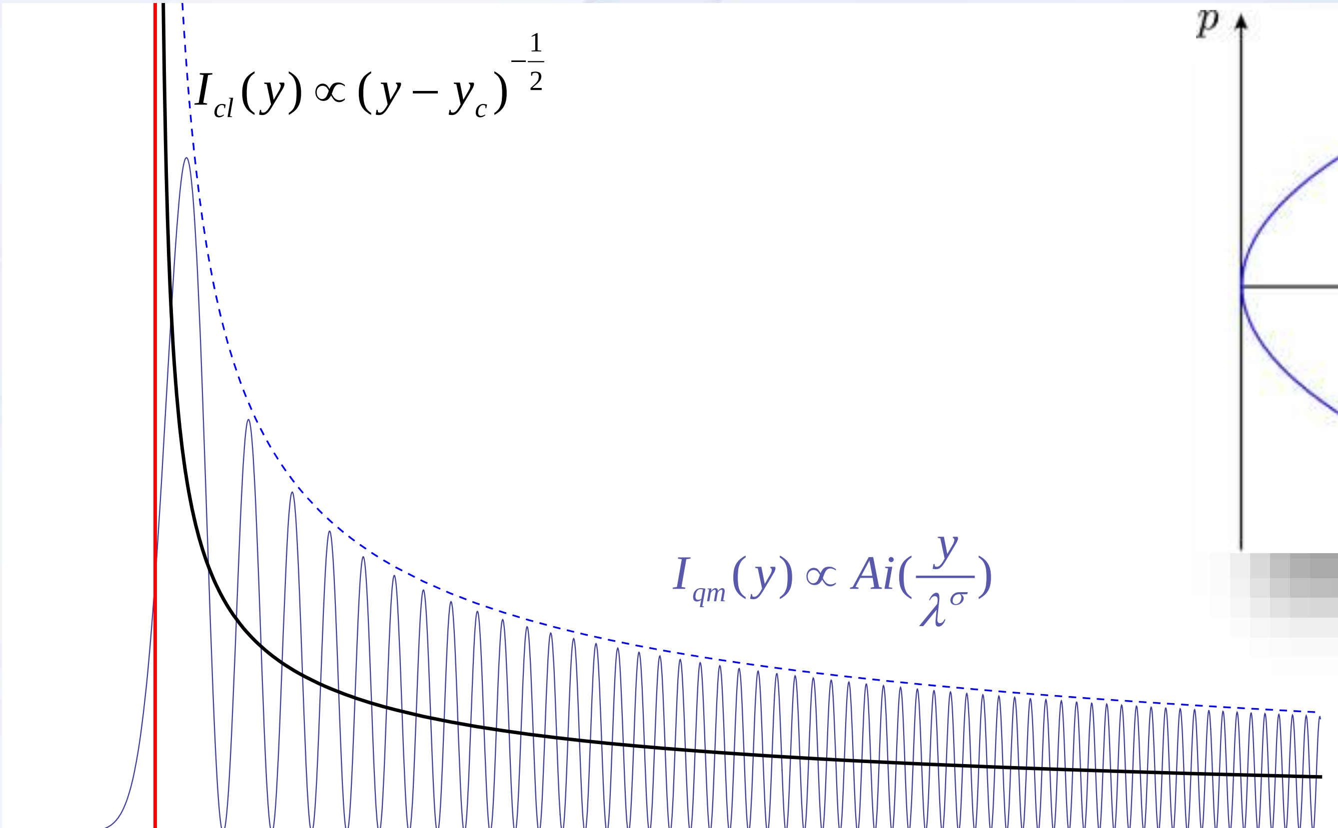


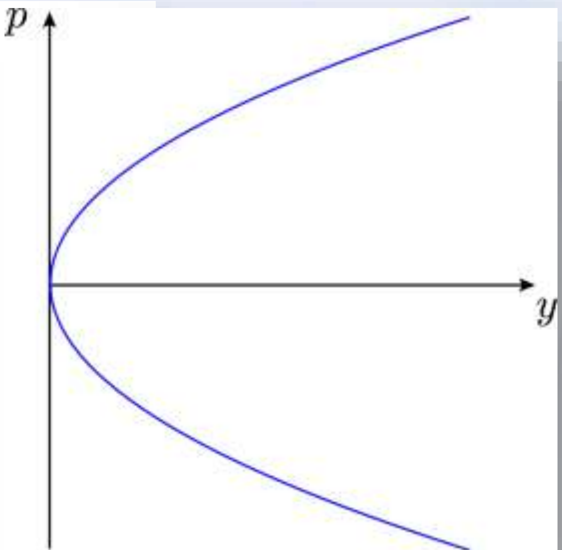
$$P(I' > I) = \int_I^\infty dI'\, p(I') \propto I^{-2}$$

Classical/Ray-Intensity



Ray- and Wave-Intensity at a Caustic

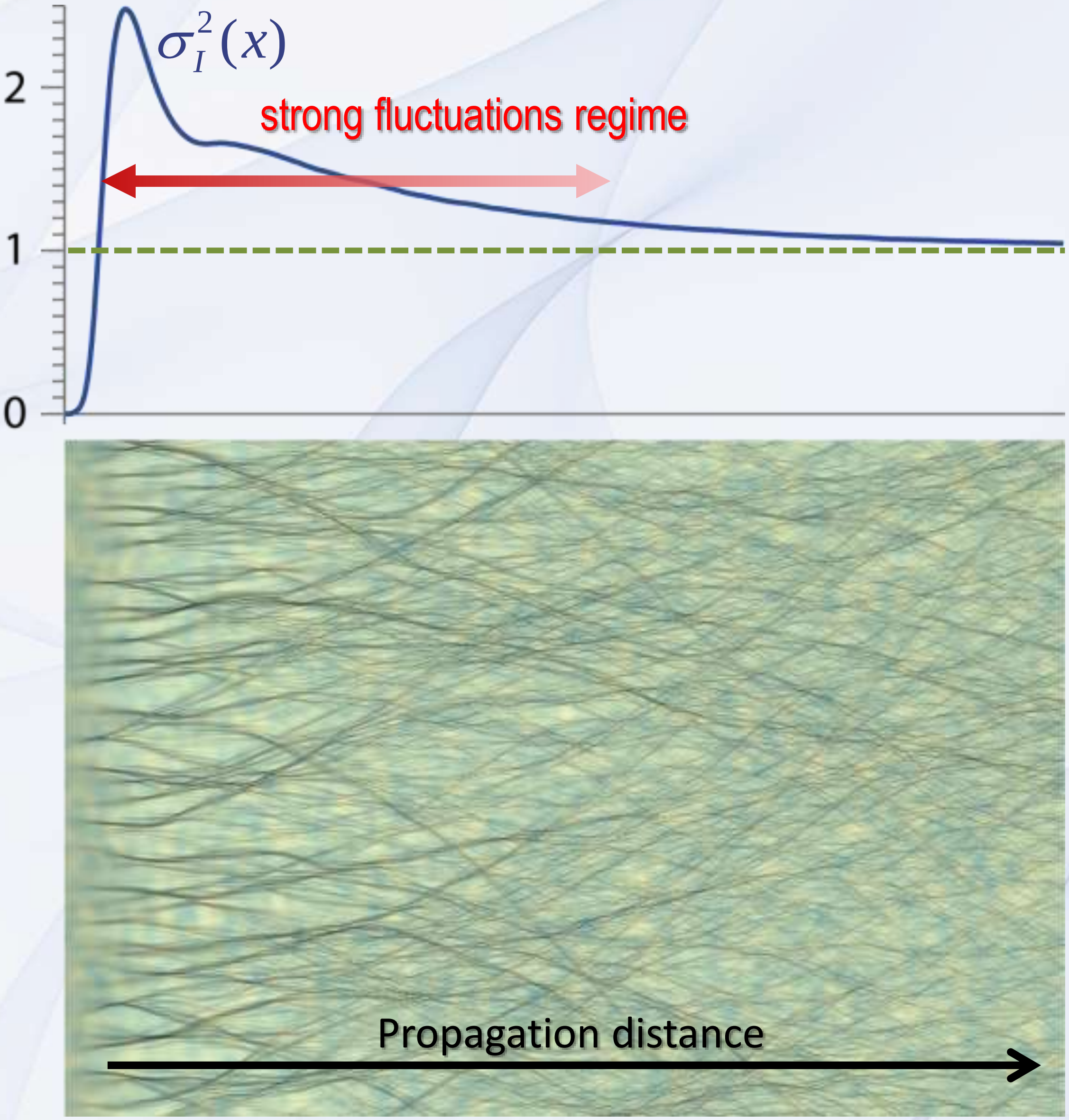




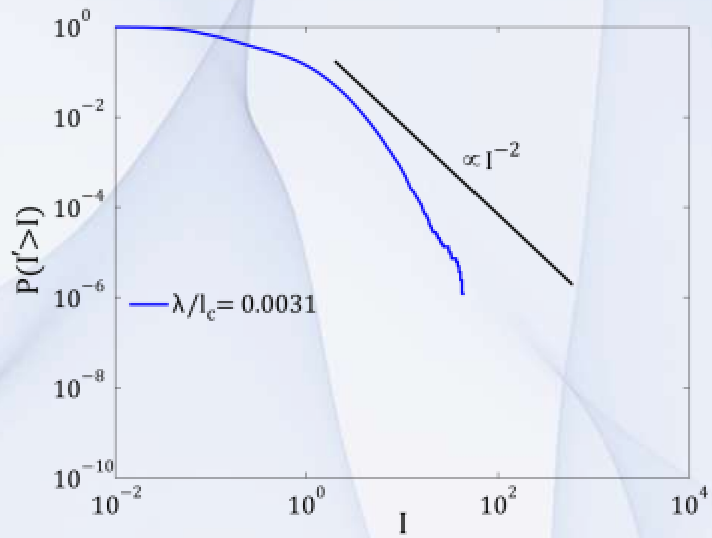
$$p(I) \propto I^{-3}$$

$$P(I' > I) = \int_I^\infty dI'\, p(I') \propto I^{-2}$$

Theory of intensity statistics



Power-law tails due to caustics only for extreme scale separation



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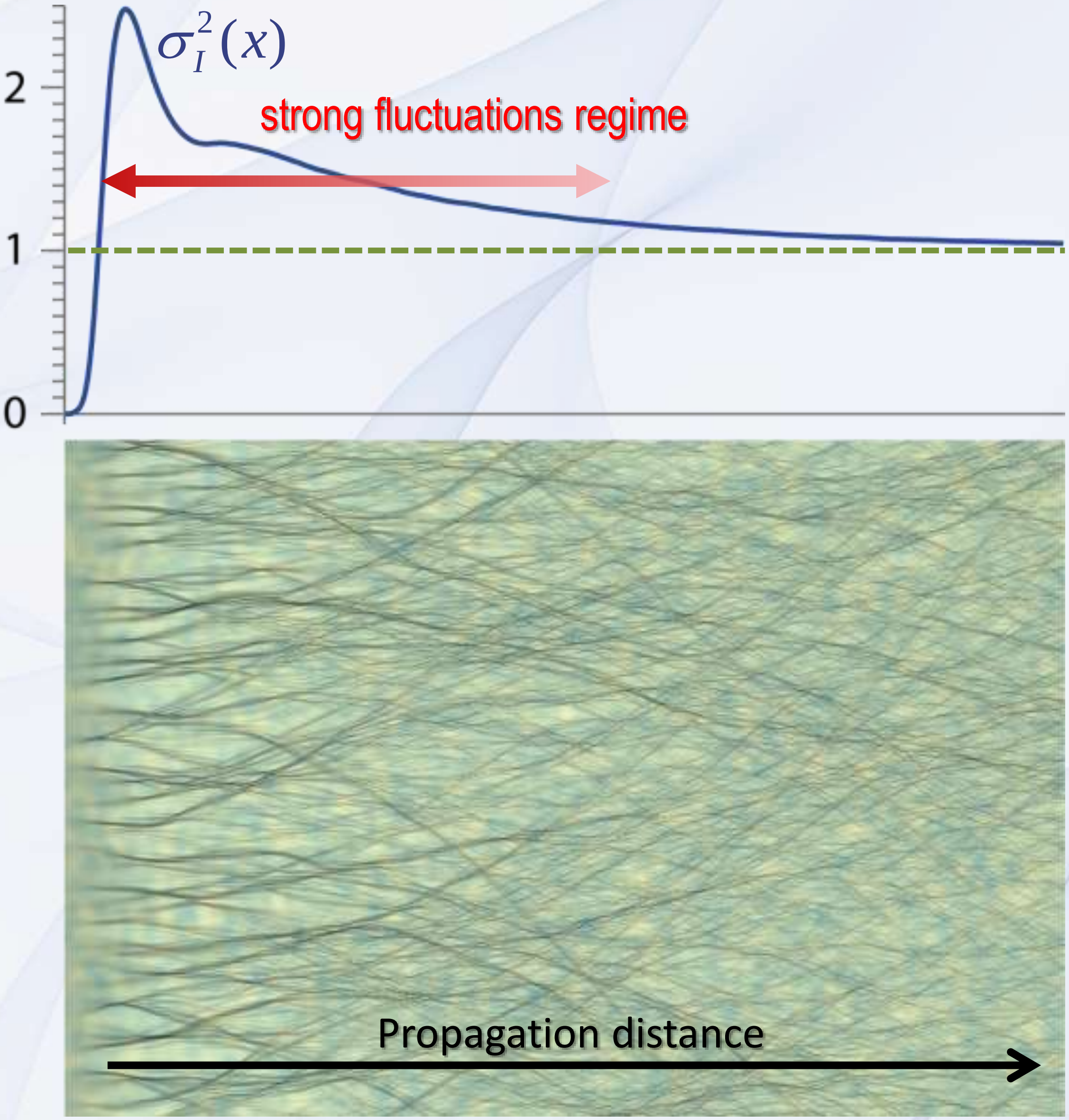

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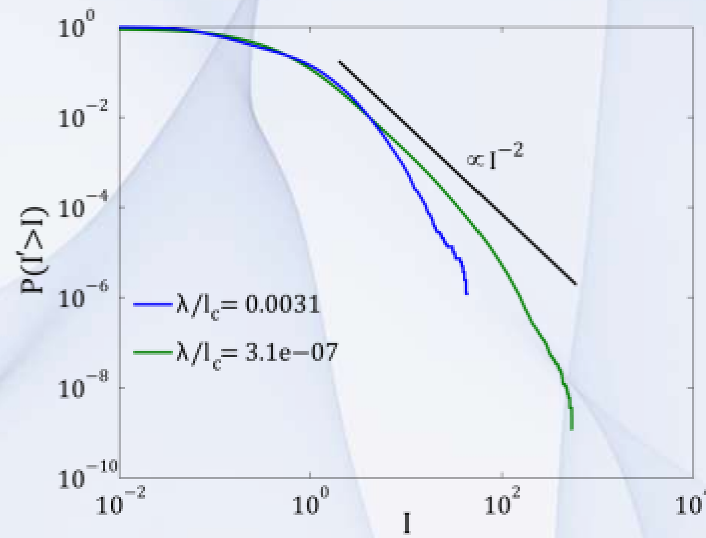

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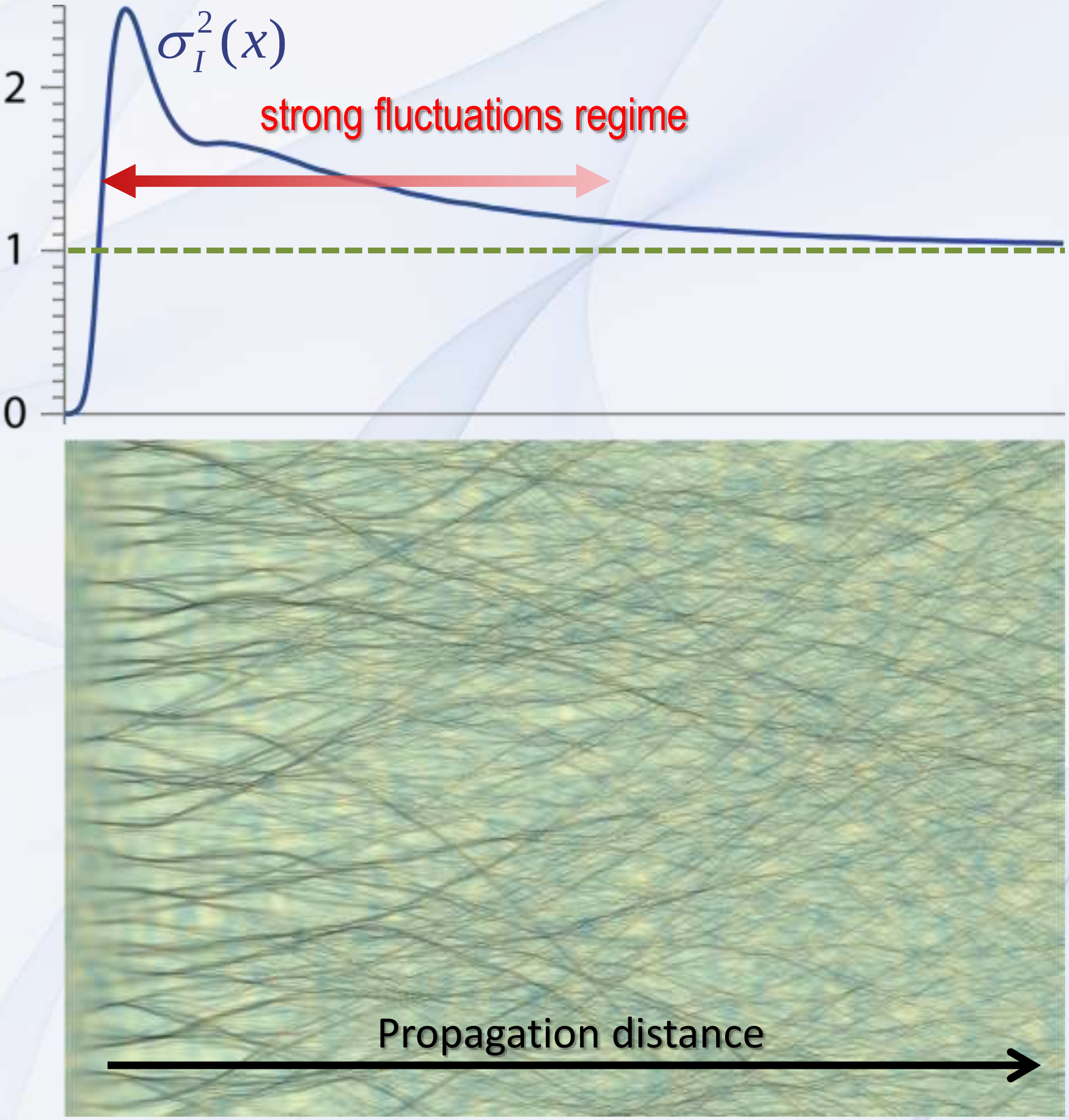
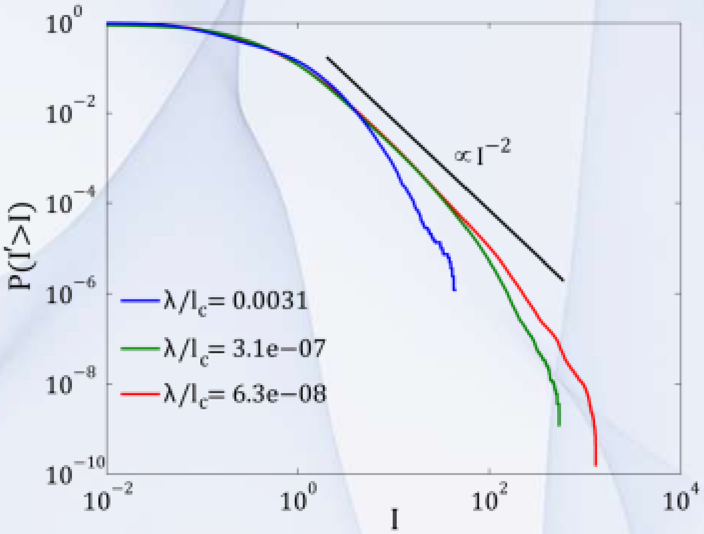
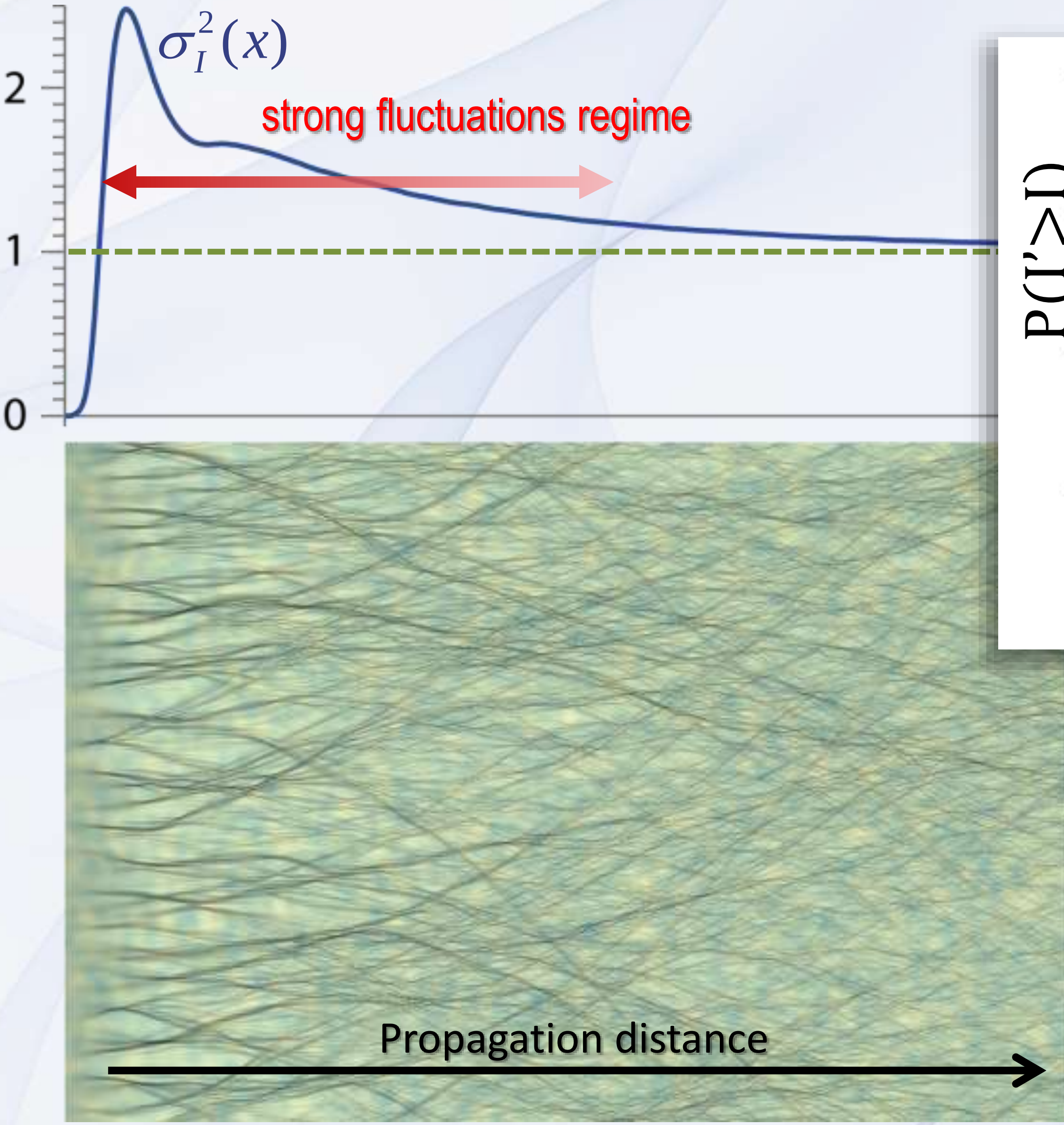


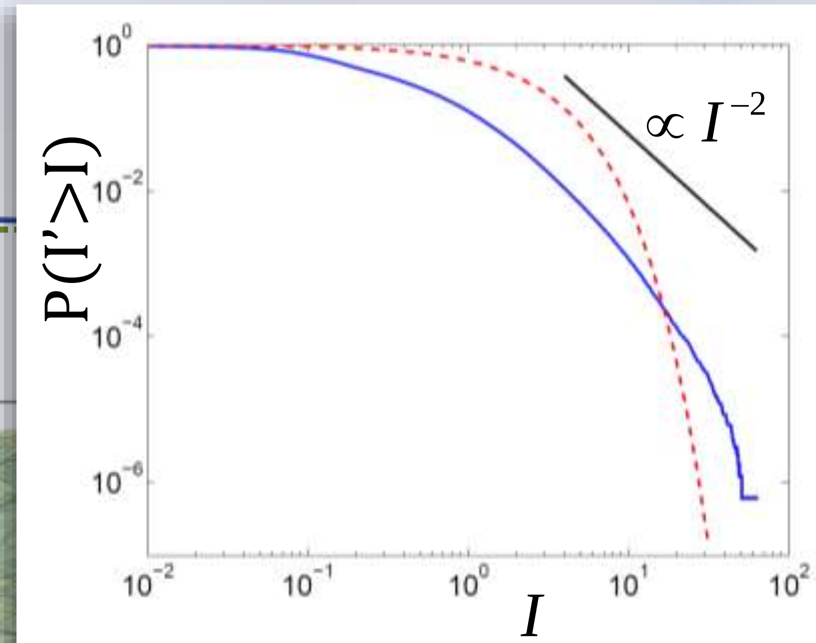
Power-law tails due to caustics only for extreme scale separation

$$\frac{\ell_c}{\lambda} \gg 10^6$$



Theory of intensity statistics





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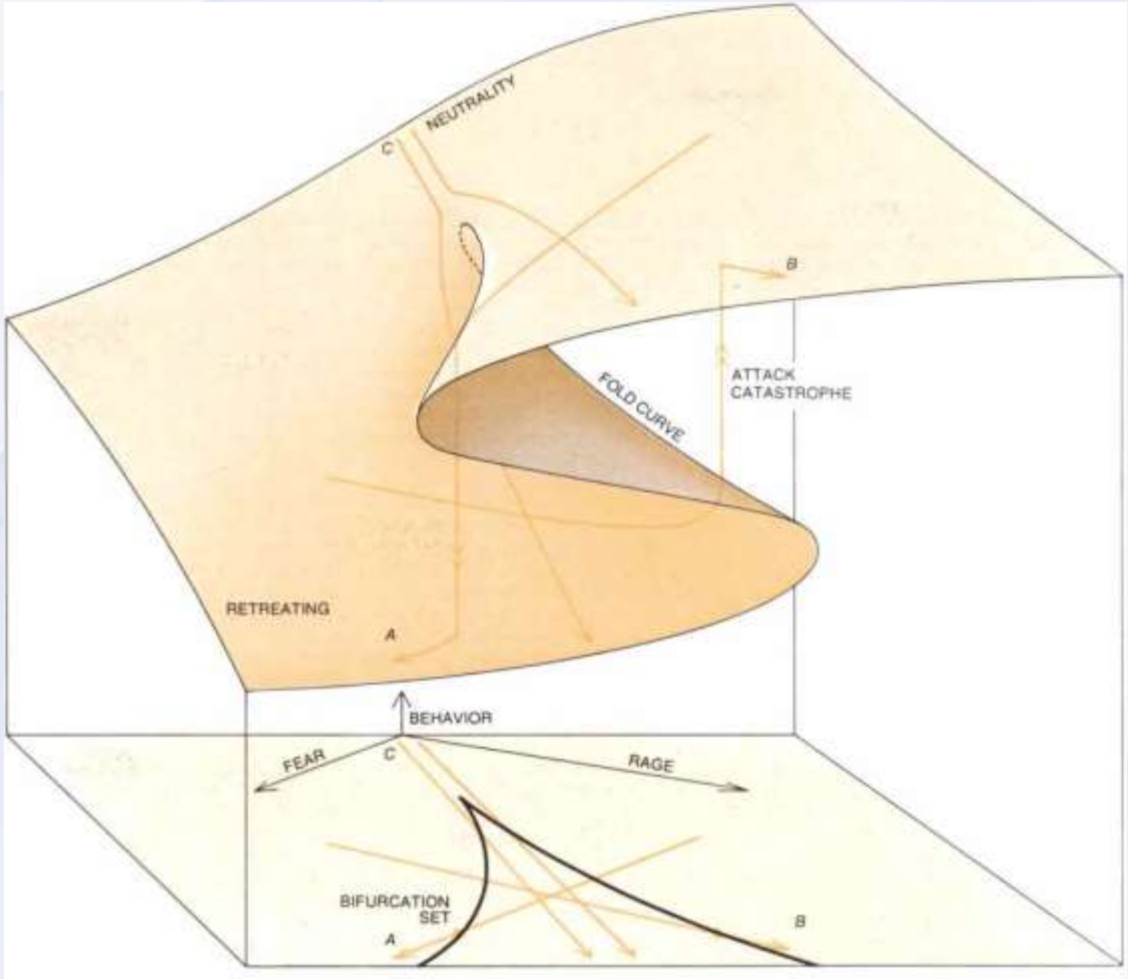
Catastrophe theory

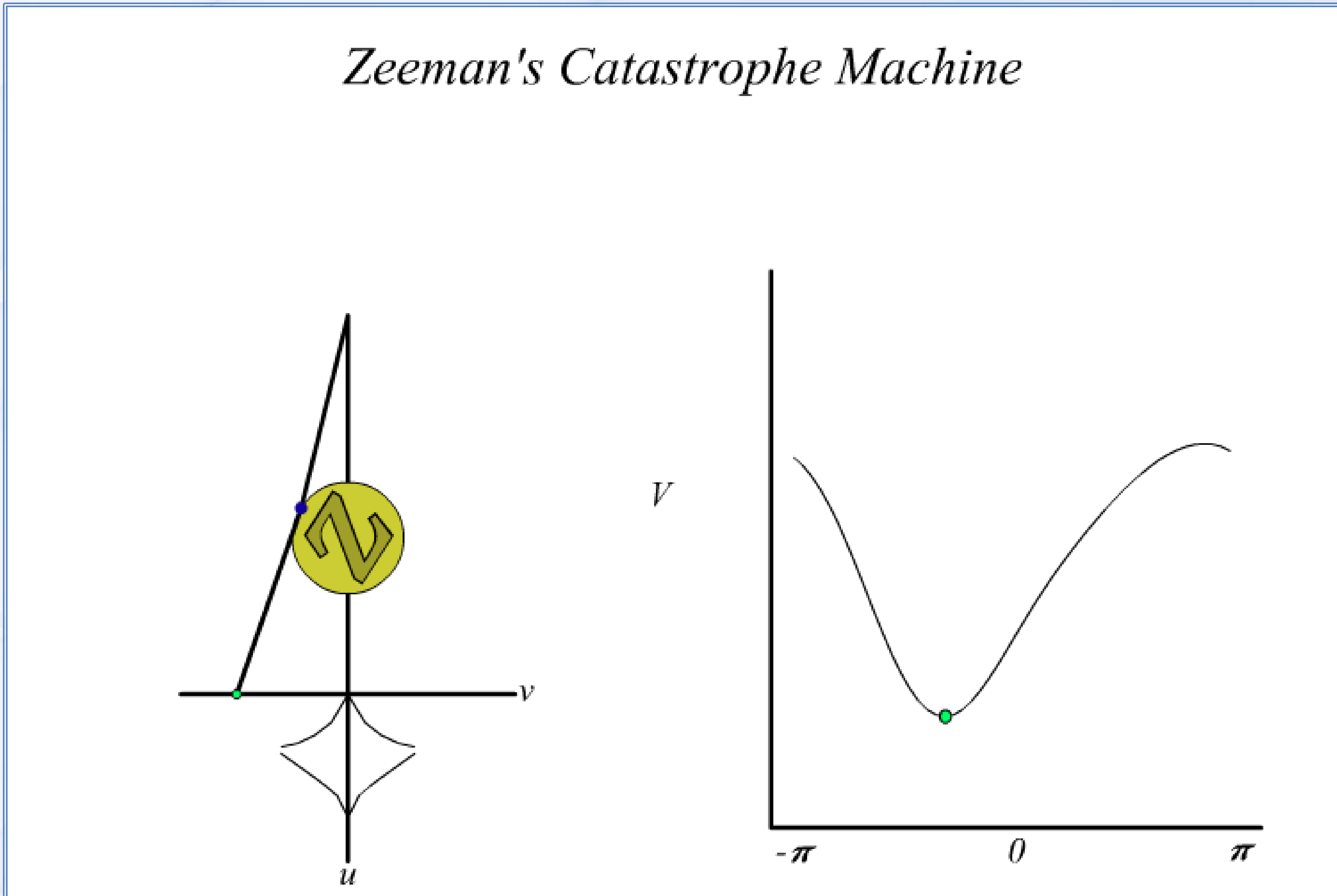
- Bifurcations from a different perspective.
- Studying singularities (discontinuities) of gradient maps $\nabla\Phi = 0$
- Thom, Arnold & Zeemann
- Most important result: classification theorem (Thom's Theorem)

$$\text{Potential/Lyapunov-Function} \quad \Phi(\vec{S},\vec{C})$$

The cusp catastrophe



Zeeman, E.C., 1976. Catastrophe Theory. Scientific American 65–83.
doi:10.1038/scientificamerican0476-65



Zeeman's Catastrophe Machine in Flash by Daniel J. Cross
http://lagrange.physics.drexel.edu/flash/zcm/

Classification

CATASTROPHE		CONTROL DIMENSIONS	BEHAVIOR DIMENSIONS	FUNCTION	FIRST DERIVATIVE
CUSPOIDS	FOLD	1	1	$\frac{1}{3}x^3 - ax$	$x^2 - a$
	CUSP	2	1	$\frac{1}{4}x^4 - ax - \frac{1}{2}bx^2$	$x^3 - a - bx$
	SWALLOWTAIL	3	1	$\frac{1}{5}x^5 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3$	$x^4 - a - bx - cx^2$
	BUTTERFLY	4	1	$\frac{1}{6}x^6 - ax - \frac{1}{2}bx^2 - \frac{1}{3}cx^3 - \frac{1}{4}dx^4$	$x^5 - a - bx - cx^2 - dx^3$
UMBILICS	HYPERBOLIC	3	2	$x^3 + y^3 + ax + by + cxy$	$3x^2 + a + cy$ $3y^2 + b + cx$
	ELLIPTIC	3	2	$x^3 - xy^2 + ax + by + cx^2 + cy^2$	$3x^2 - y^2 + a + 2cx$ $-2xy + b + 2cy$
	PARABOLIC	4	2	$x^2y + y^4 + ax + by + cx^2 + dy^2$	$2xy + a + 2cx$ $x^2 + 4y^3 + b + 2dy$

SEVEN ELEMENTARY CATASTROPHES describe all possible discontinuities in phenomena controlled by no more than four factors. Each of the catastrophes is associated with a potential function in which the control parameters are represented as coefficients (*a, b, c, d*) and the behavior of the system is determined by the variables (*x, y*). The behavior surface in each catastrophe model is the graph of all the points where the first derivative of this function is equal to zero or, when there are two first derivatives, where both are equal to zero.

Zeeman, E.C., 1976. Catastrophe Theory. Scientific American 65–83.

Example: the cusp

Example: the fold

Catastrophe optics

$$\Phi(s,C) = -\ell(s,X,Y)$$

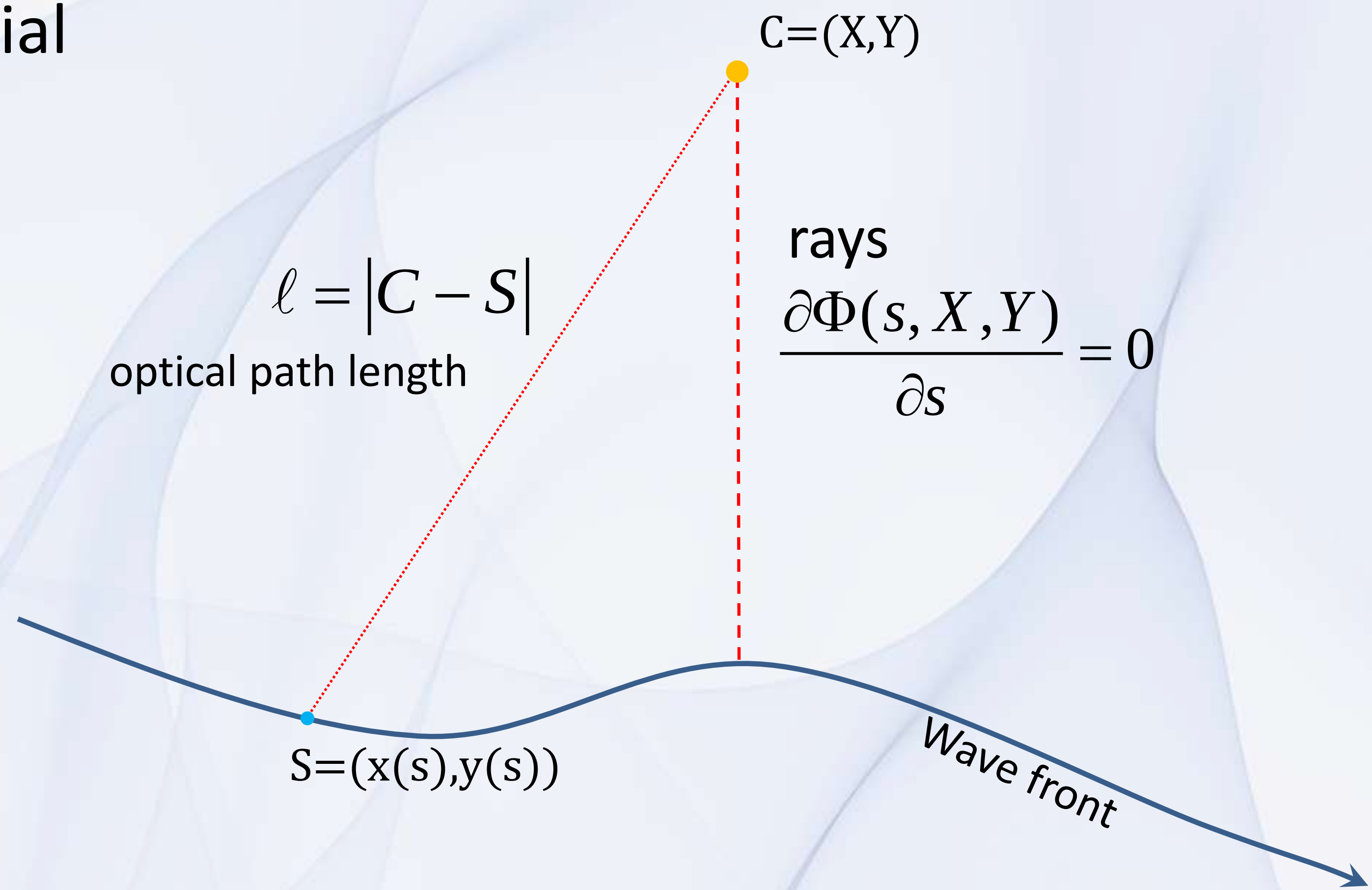
Potential

$$\ell = |C - S|$$

optical path length

rays

$$\frac{\partial \Phi(s,X,Y)}{\partial s} = 0$$



Berry & Upstill 1980
J. F. Nye, *Natural Focusing and Fine Structure of Light: Caustics and Wave Dislocations*, IOP, 1999

Caustics

Caustics are the catastrophes of ray optics with $0 = \dfrac{\partial^2 \Phi}{\partial s^2} = \dfrac{\partial^3 \Phi}{\partial s^3} = \ldots$

Standard polynomials Φ for the elementary catastrophes with codimension $K \leq 4$

Name	Symbol	K	$\Phi(s; C)$
fold	A_2	1	$s^3/3 + Cs$
cusp	A_3	2	$s^4/4 + C_2 s^2/2 + C_1 s$
swallowtail	A_4	3	$s^5/5 + C_3 s^3/3 + C_2 s^2/2 + C_1 s$
elliptic umbilic	D_4^-	3	$s_1^3 - 3 s_1 s_2^2 - C_3(s_1^2 + s_2^2) - C_2 s_2 - C_1 s_1$
hyperbolic umbilic	D_4^+	3	$s_1^3 + s_2^3 - C_3 s_1 s_2 - C_2 s_2 - C_1 s_1$
butterfly	A_5	4	$s^6/6 + C_4 s^4/4 + C_3 s^3/3 + C_2 s^2/2 + C_1 s_1$
parabolic umbilic	D_5	4	$s_1^4 + s_1 s_2^2 + C_4 s_2^2 + C_3 s_1^2 + C_2 s_2 + C_1 s_1$

Berry & Upstill, 1980

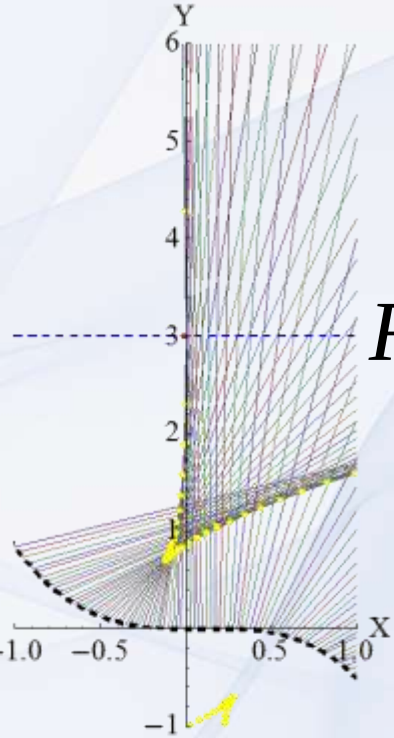
[Example cusp]

[Example fold]

Heller europhys.news 2005

"...nothing more catastrophic than the loss of two rays occurs. Nevertheless, the word catastrophe has been retained."
J.F. Nye in *Natural focusing and the fine structure of light*.

Wave amplitude near the fold caustic: diffraction



$$\phi(x, X) = \frac{1}{3}\alpha\, x^3 + X\, x$$

normal form of the fold caustic

$$R \gg X$$

$$J(X) = \int_{-\infty}^{\infty} \mathrm{d}x e^{-ik\phi(x,X)}$$

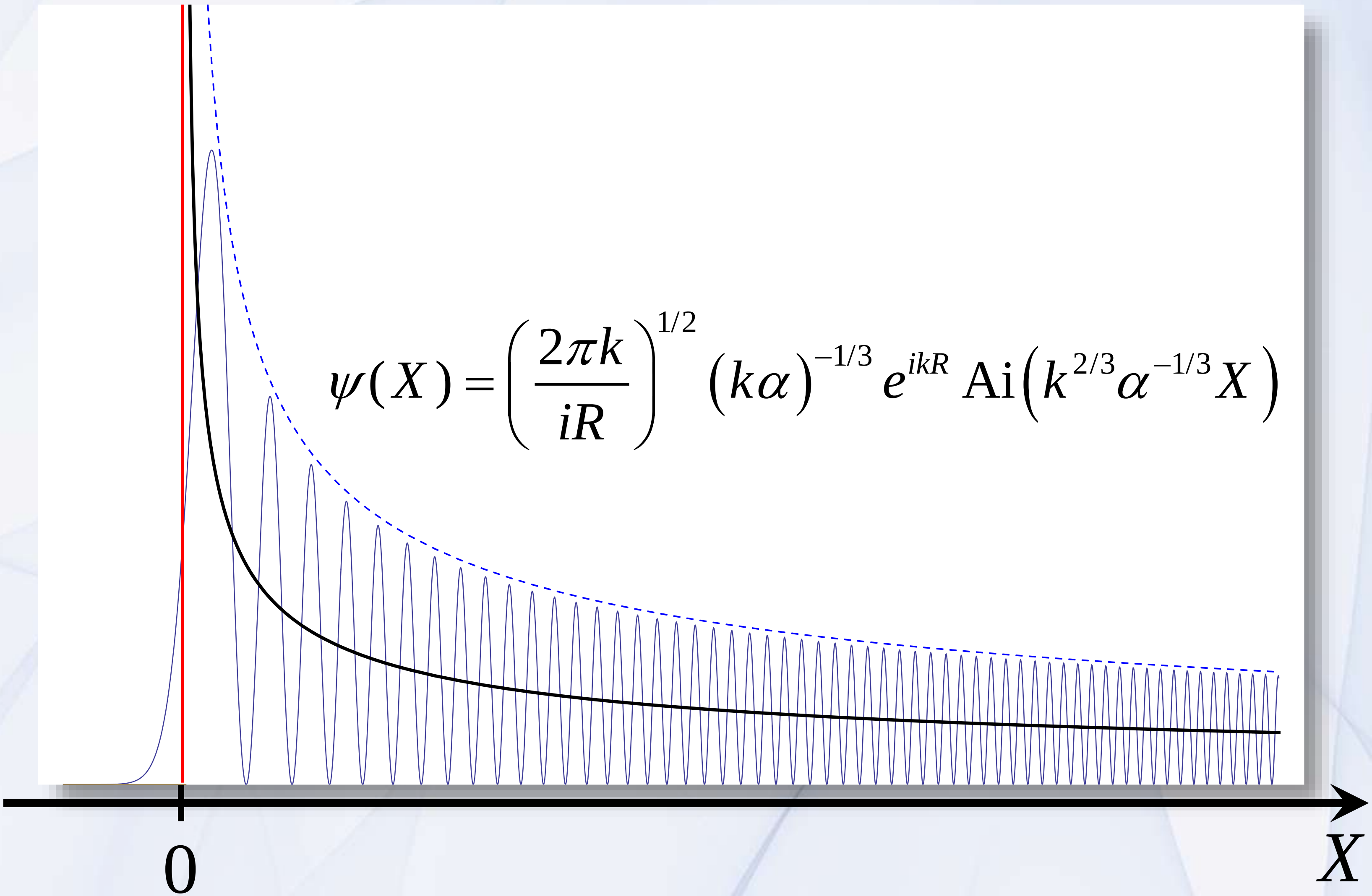
diffraction integral

$$\psi(X) \propto J(X)$$

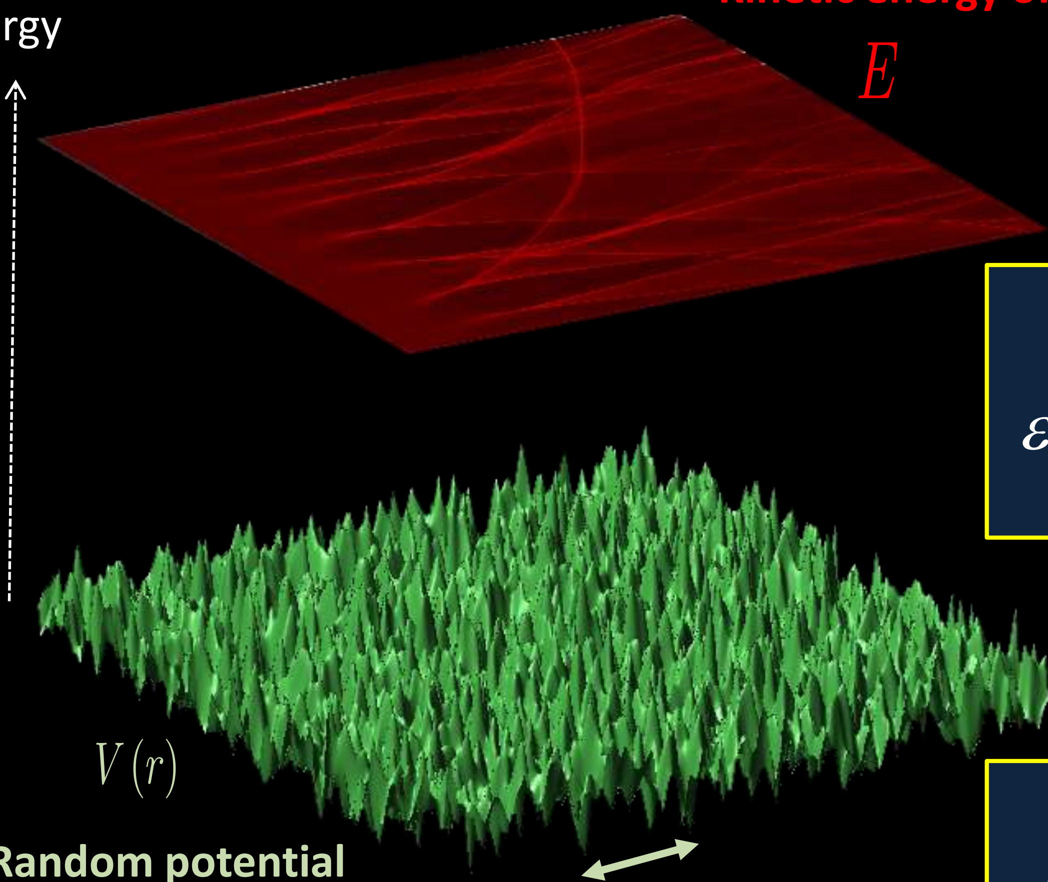
Fold at distance R from the initial wave front

$$\boxed{\psi(X) = \left(\frac{2\pi k}{iR}\right)^{1/2} (k\alpha)^{-1/3} e^{ikR}\, \mathrm{Ai}\left(k^{2/3}\alpha^{-1/3} X\right)}$$

Wave amplitude near the fold caustic: diffraction



Reminder: characteristics of the random medium



$$\varepsilon = \frac{\sqrt{\left\langle V(r)^2 \right\rangle - \left\langle V(r) \right\rangle^2}}{E} \ll 1$$

$$\lambda \leq \ell_c$$

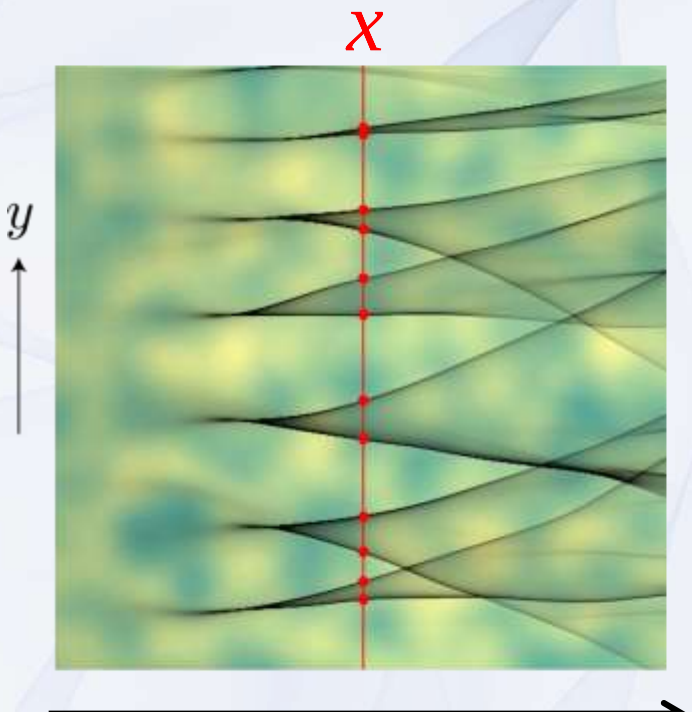
ℓ_c = correlation length, i.e. typical length scale

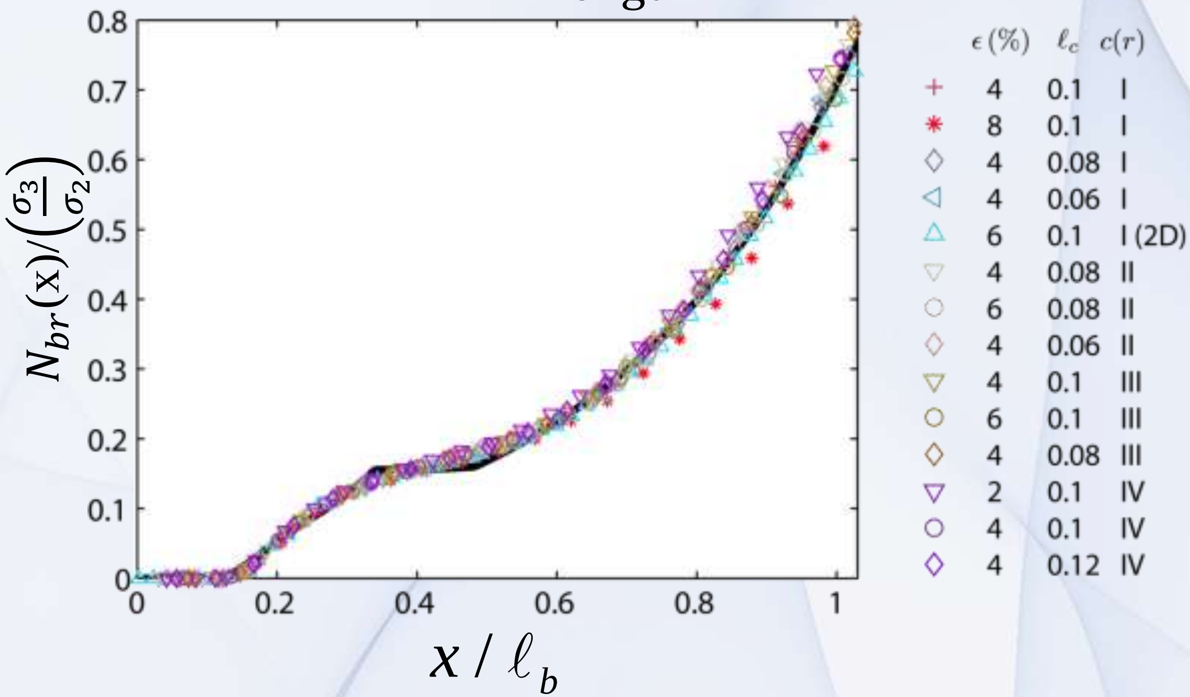
Reminder: characteristic length scale

$\ell_c \gg \lambda$
$(\lambda = \text{wavelength})$

branching length $\boxed{\ell_b \propto \ell_c \varepsilon^{-2/3}} \underset{\varepsilon \ll 1}{\ll} \boxed{\ell_{\text{mfp}} \propto \ell_c \varepsilon^{-2}}$ mean free path

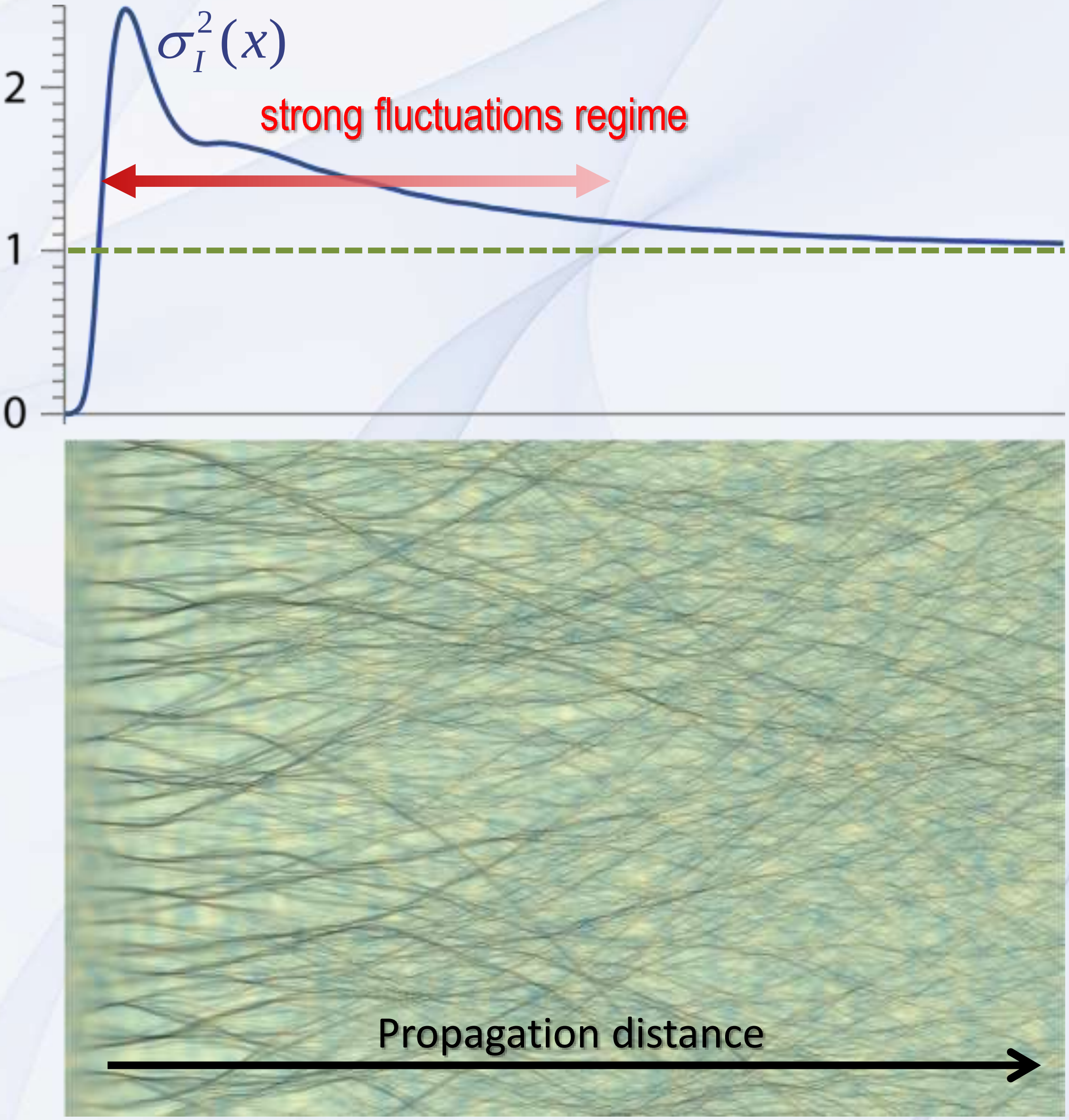
$$N_{br}(x) = \frac{\#\text{Caustics}}{2 \cdot \text{Length}}$$





J.J. Metzger, R. Fleischmann, and T. Geisel, Phys. Rev. Lett. **105** (2010).

Theory of intensity statistics

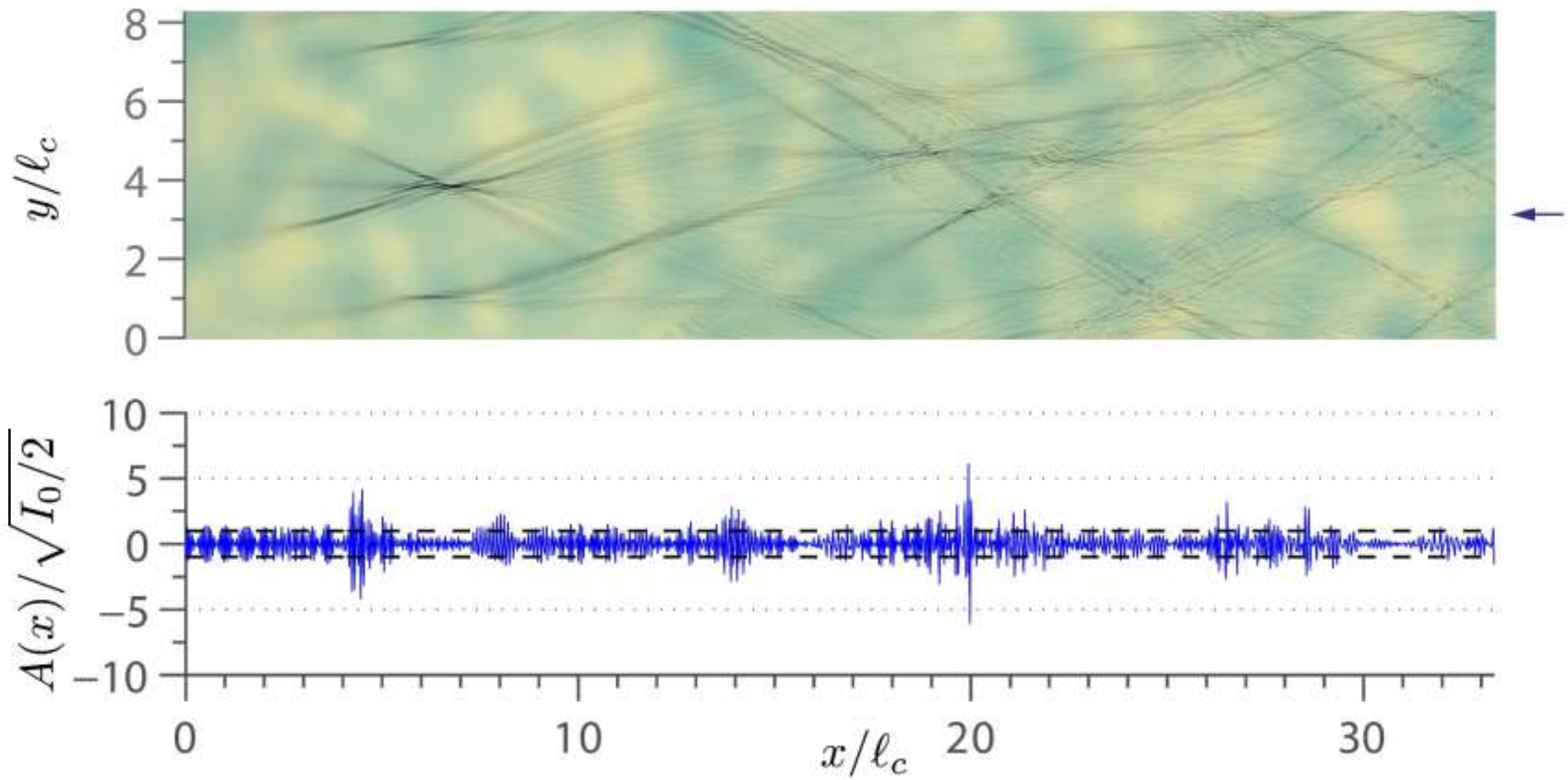


- power-law tails only in semi-classical limit (not realistic)
- intensity distribution still unknown but

  Analytic result for the distribution of extreme (rogue) waves

J. J. Metzger, R.Fleischmann, and T. Geisel, PRL 112, 203903 (2014).

The statistics of extreme (rogue) waves



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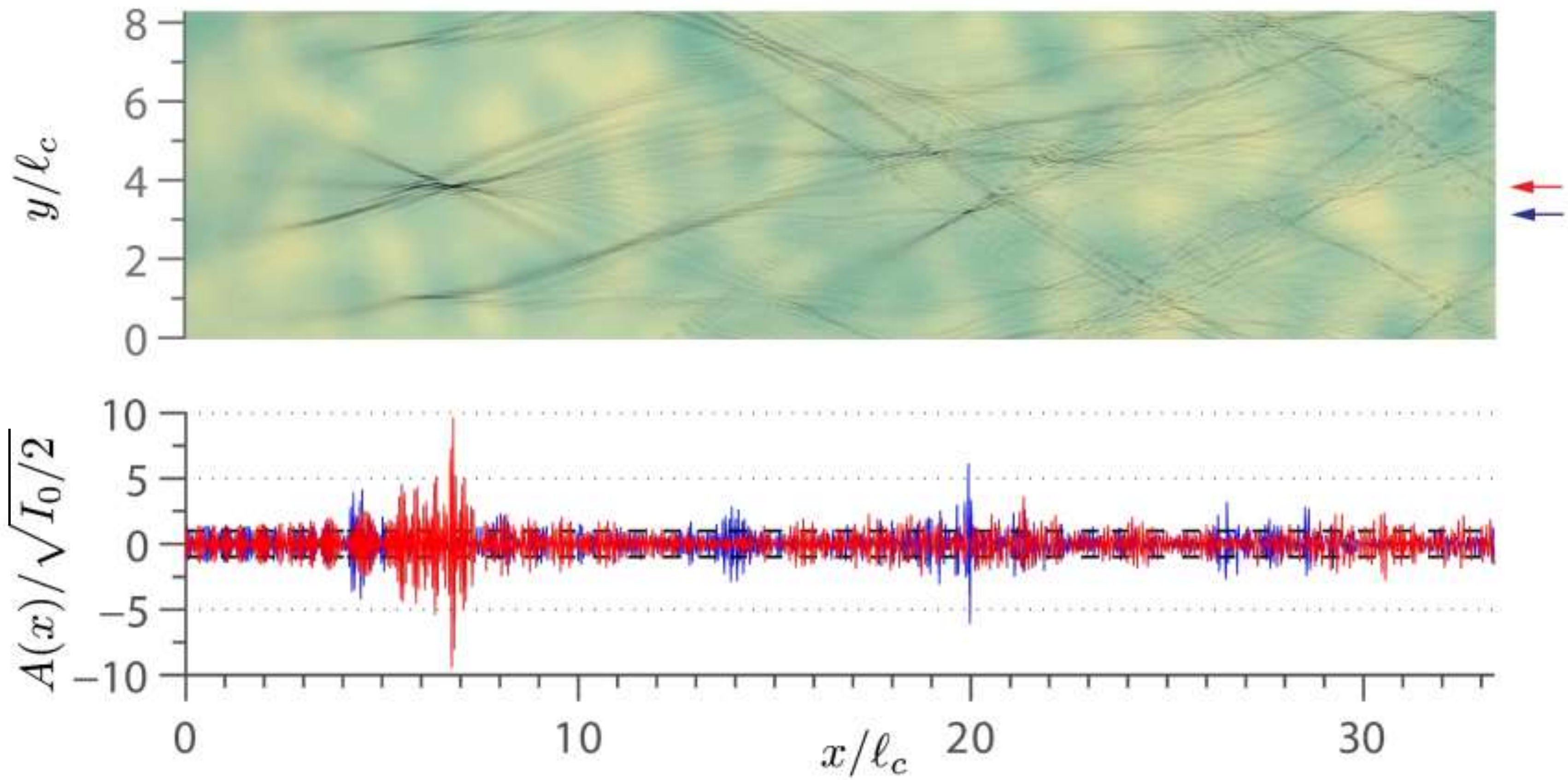

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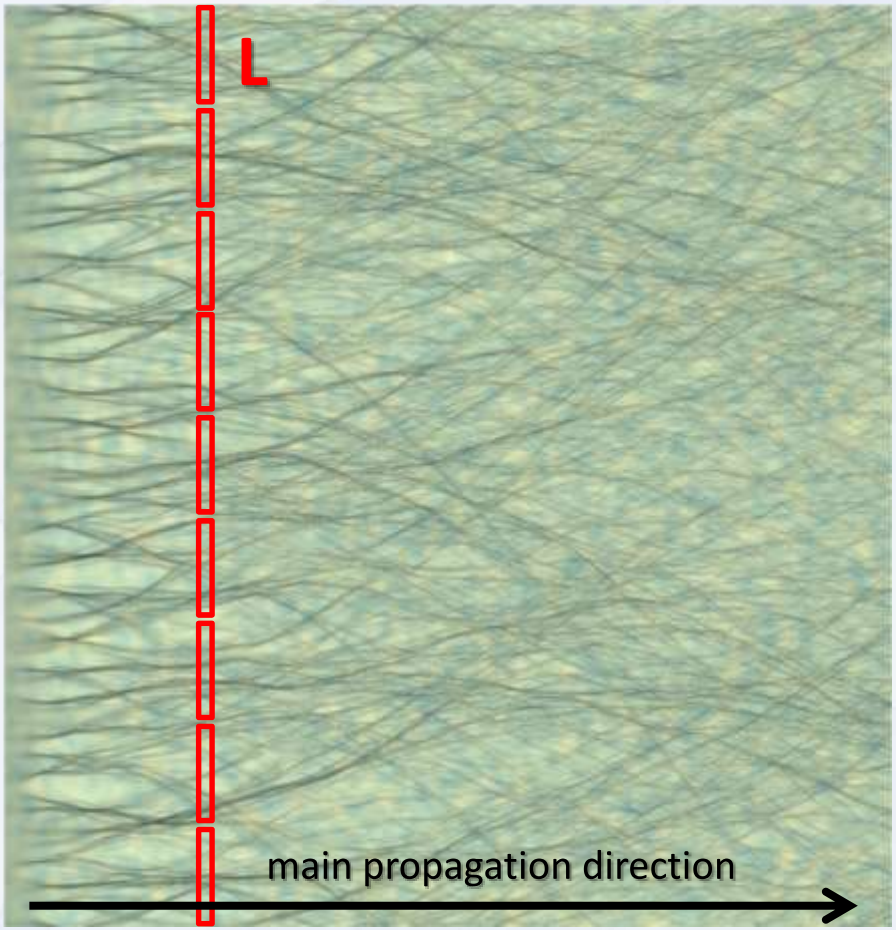
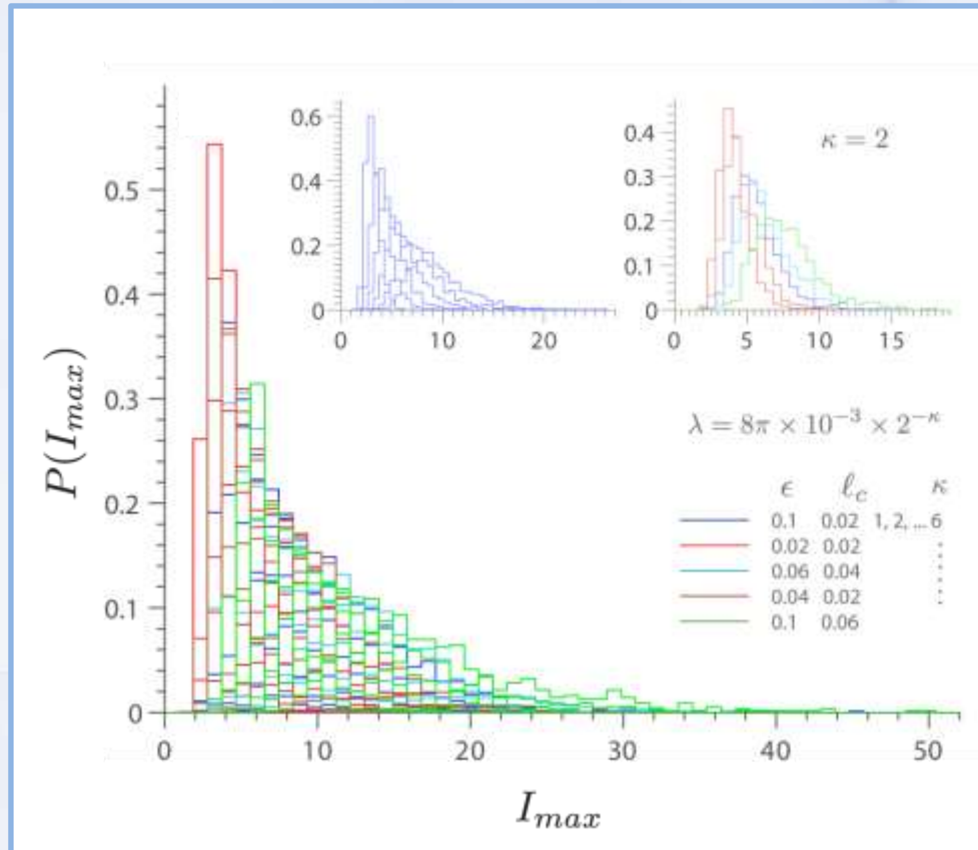
J. J. Metzger, R.Fleischmann, and T. Geisel, PRL 112 (2014)

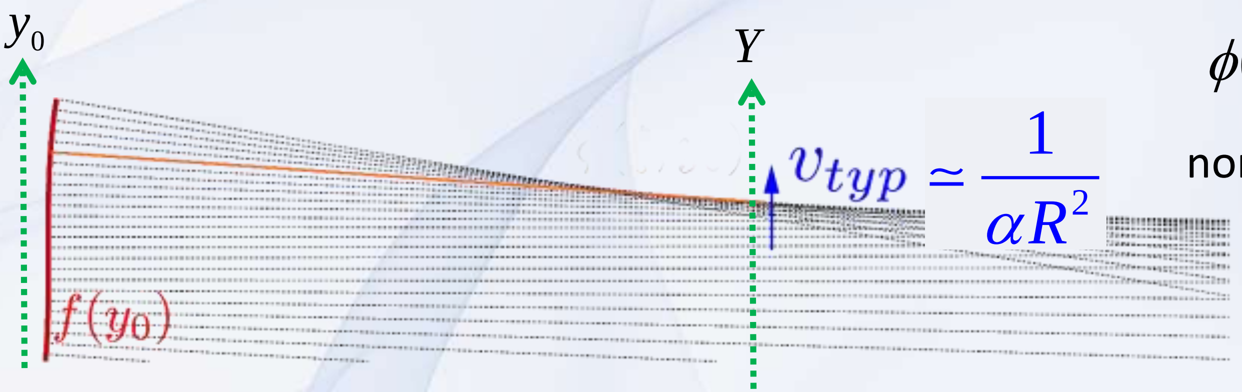




The statistics of extreme (rogue) waves

J. J. Metzger, R.Fleischmann, and T. Geisel, PRL 112 (2014)

(see e.g. J. F. Nye, *Natural Focusing and Fine Structure of Light: Caustics and Wave Dislocations*, IOP Publishing, 1999).



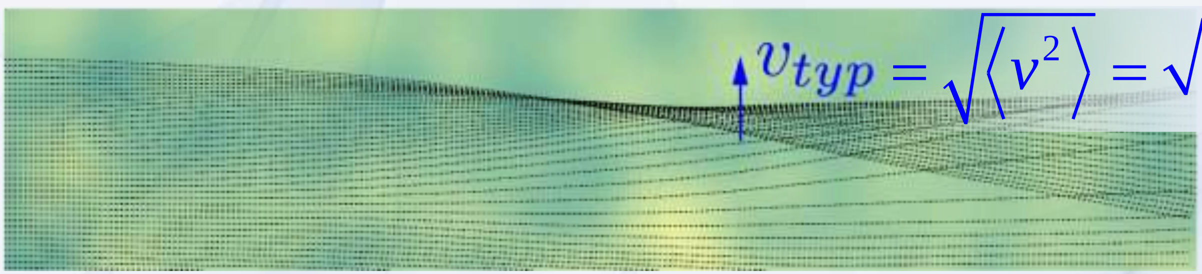
$$v_{typ} \simeq \frac{1}{\alpha R^2}$$

$$f(y) = \frac{y^2}{2R} + \frac{\alpha}{3} y^3$$

$$\phi(y_0, Y) = \frac{1}{3}\alpha\, y_0^3 + Y\, y_0$$

normal form of the fold caustic

$$R \to \ell_b$$



$$v_{typ} = \sqrt{\left\langle v^2 \right\rangle} = \sqrt{\sigma^2 t}$$

$$\sqrt{\sigma_1{}^2 t_c} = \sqrt{\frac{\varepsilon^2}{\ell_c}\frac{\ell_c}{\varepsilon^{2/3}}} = \varepsilon^{2/3}$$

Mapping branched flow on catastrophe optics

$$\psi(X) = \left(\frac{2\pi k}{iR}\right)^{1/2} (k\alpha)^{-1/3} e^{ikR}\, \mathrm{Ai}\left(k^{2/3}\alpha^{-1/3} X\right)$$

The statistics of extreme (rogue) waves

J. J. Metzger, R.Fleischmann, and T. Geisel, PRL 112 (2014)

(see e.g. J. F. Nye, *Natural Focusing and Fine Structure of Light: Caustics and Wave Dislocations*, IOP Publishing, 1999).

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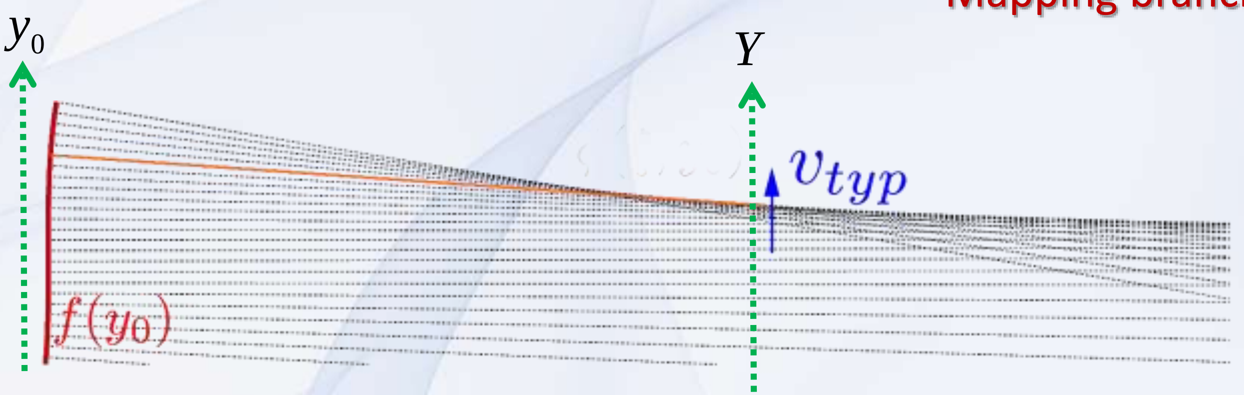

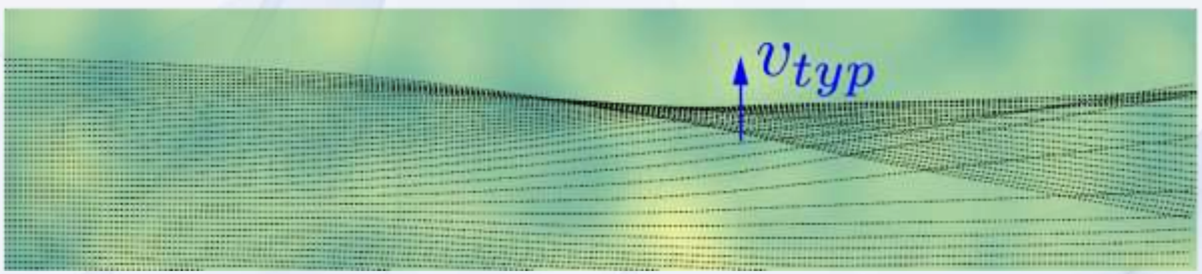

$$\alpha = \left[\ell_b^{\,2}\,\varepsilon^{2/3}\right]^{-1}$$

$$R = \ell_b$$

$$\ell_b = v_0 t_c = v_0 \ell_c \varepsilon^{-2/3}$$

$$\boxed{I_{\text{fold}} = \left|\psi^2_{\text{fold}}(k,\epsilon,\ell_c)\right| \propto (\lambda / \ell_c)^{-1/3}\epsilon^{2/9}}$$

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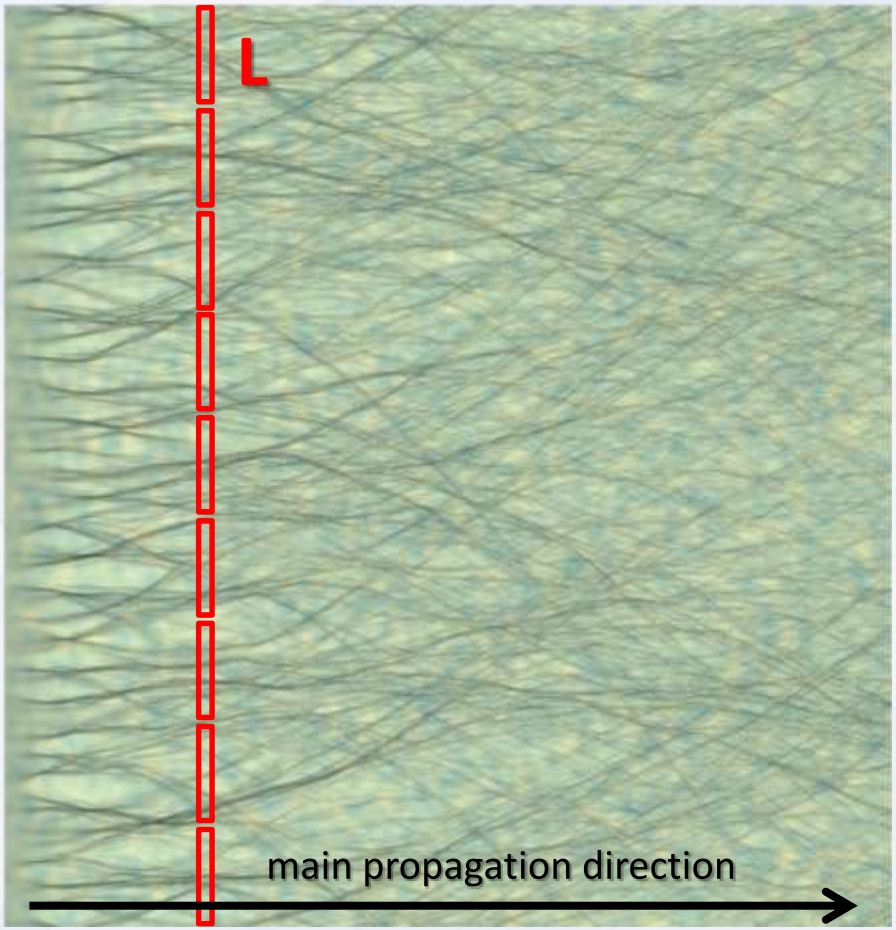
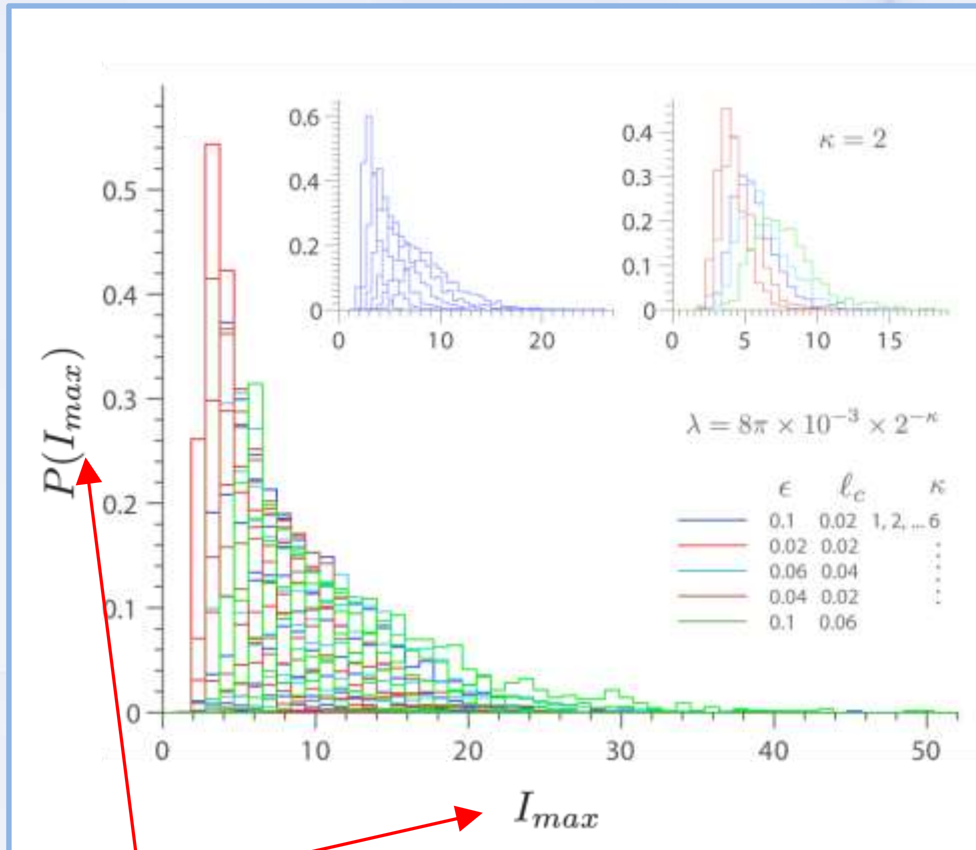
J. J. Metzger, R.Fleischmann, and T. Geisel, PRL 112 (2014)





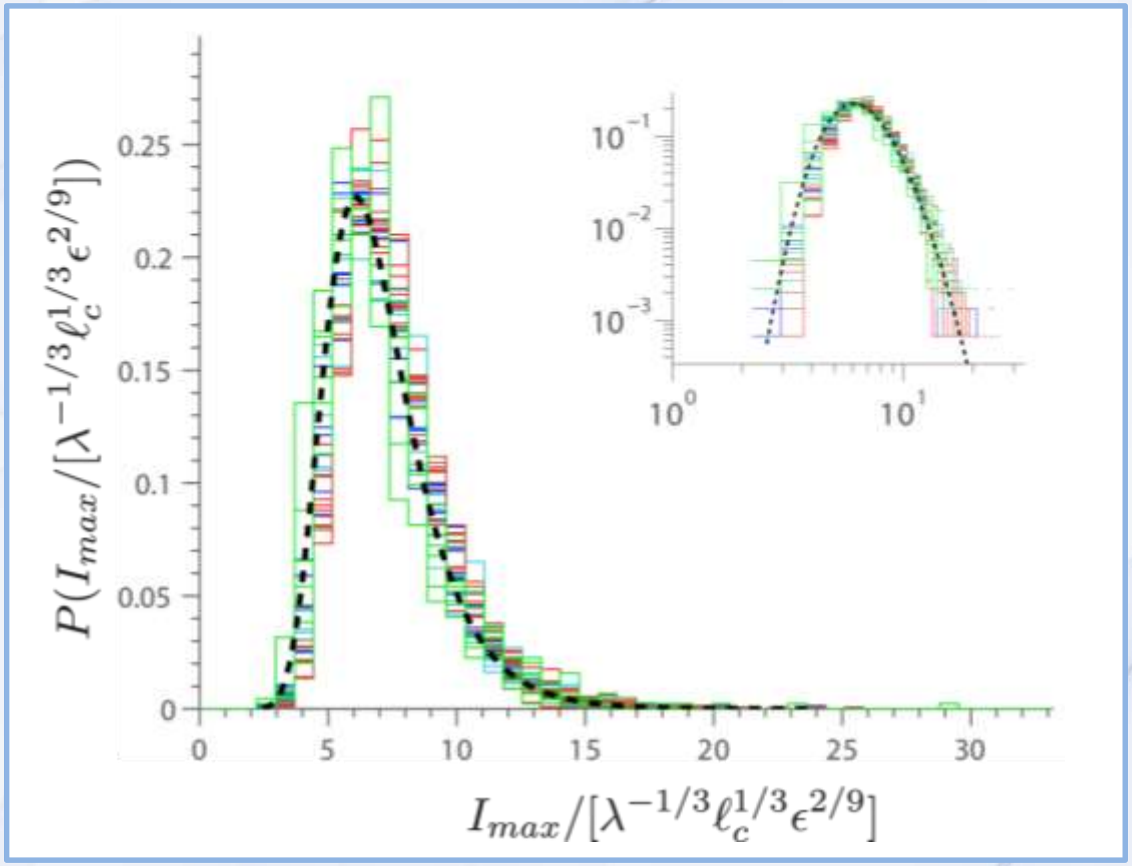
scale I_{max} with $(\lambda / \ell_c)^{-1/3} \epsilon^{2/9}$

The statistics of extreme (rogue) waves

J. J. Metzger, R.Fleischmann, and T. Geisel, PRL 112 (2014)

$$P(I/[\lambda^{-1/3}\ell_c^{1/3}\epsilon^{2/9}] \leq I_{max}) = e^{-[(I_{max}-m)/s]^{-a}}$$

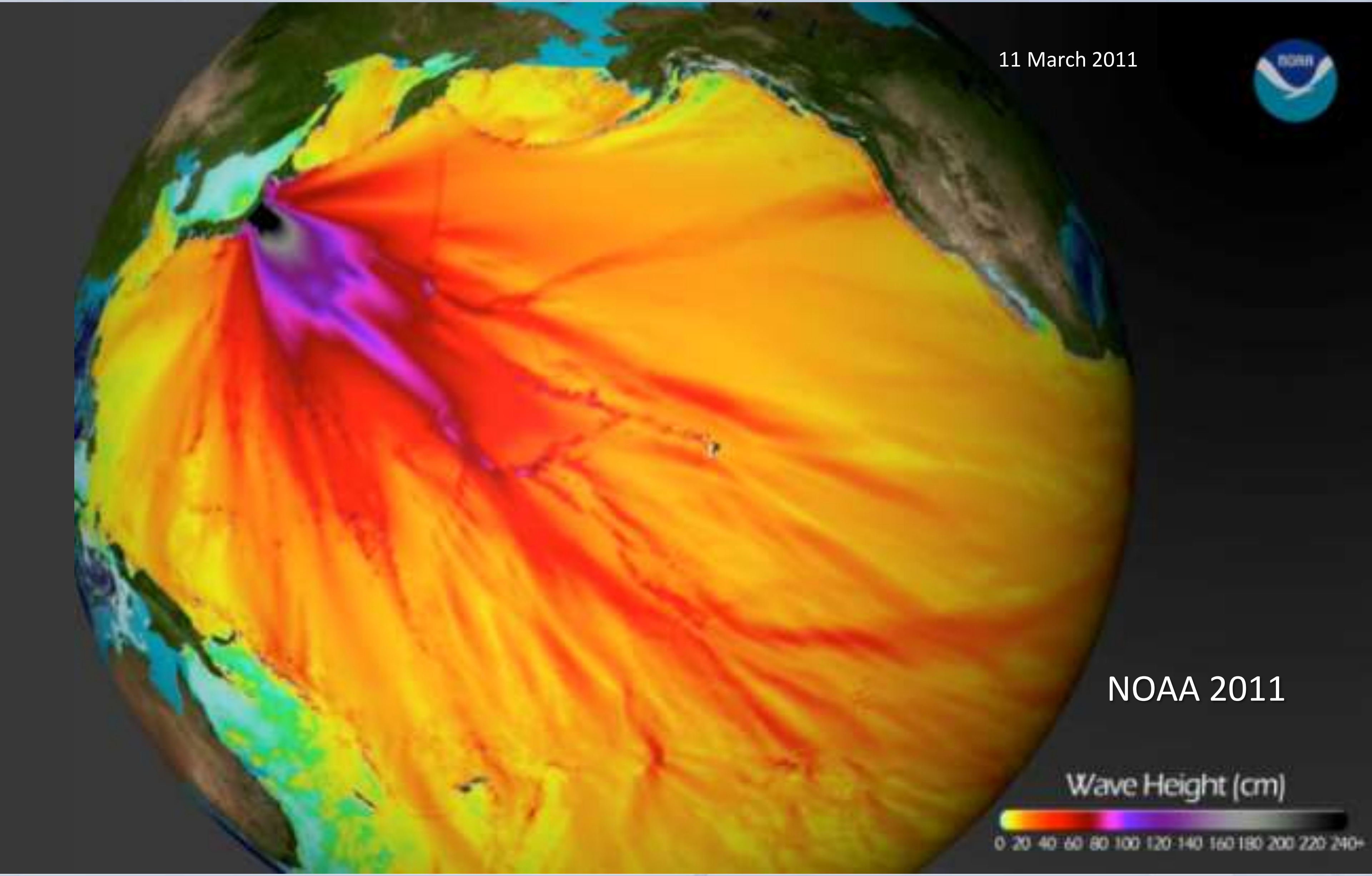
Distribution of highest waves



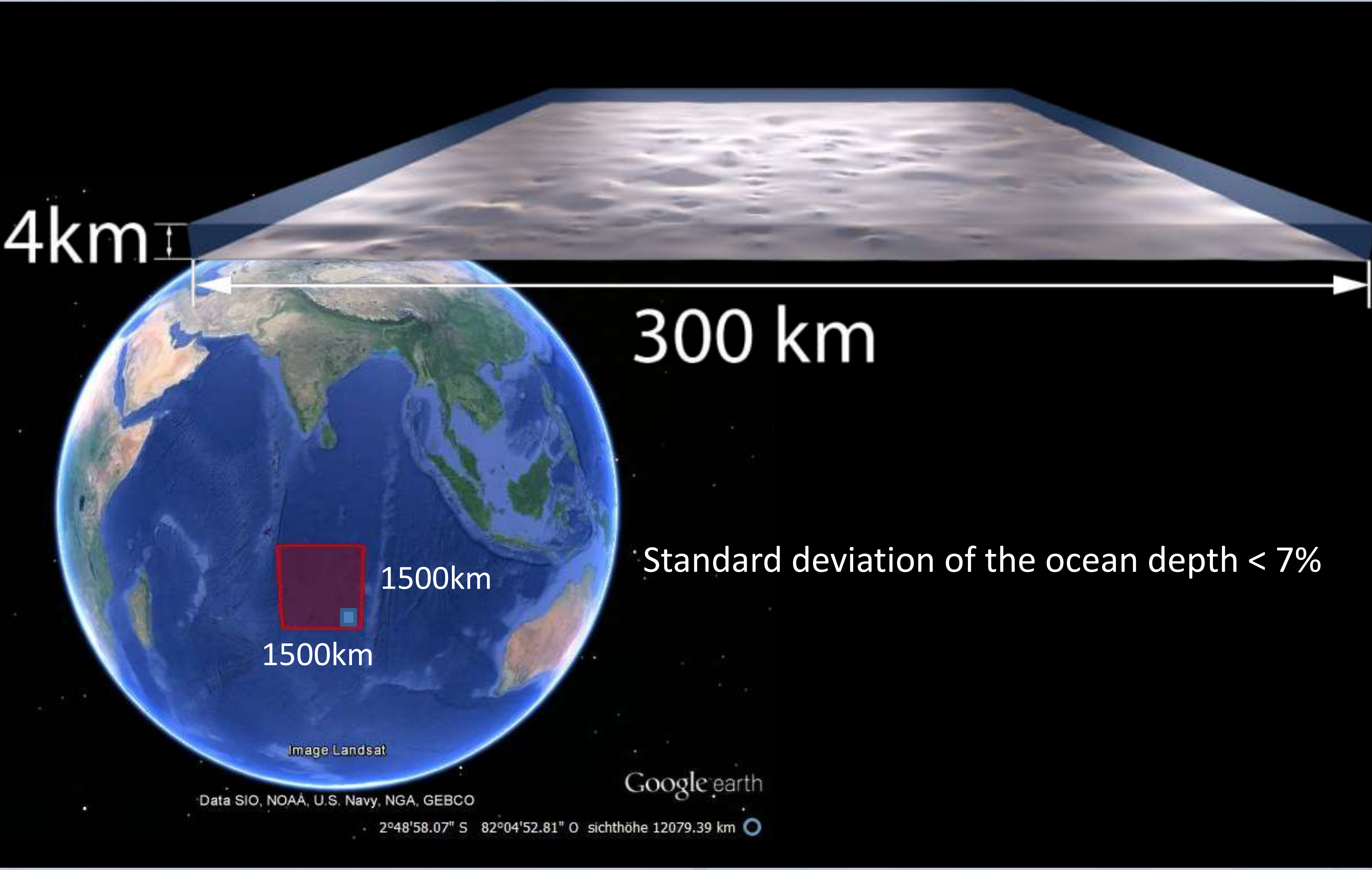
- extreme value statistics
- Stochastic nonlinear dynamics
- diffraction
- interference

$\lambda =$ wavelength

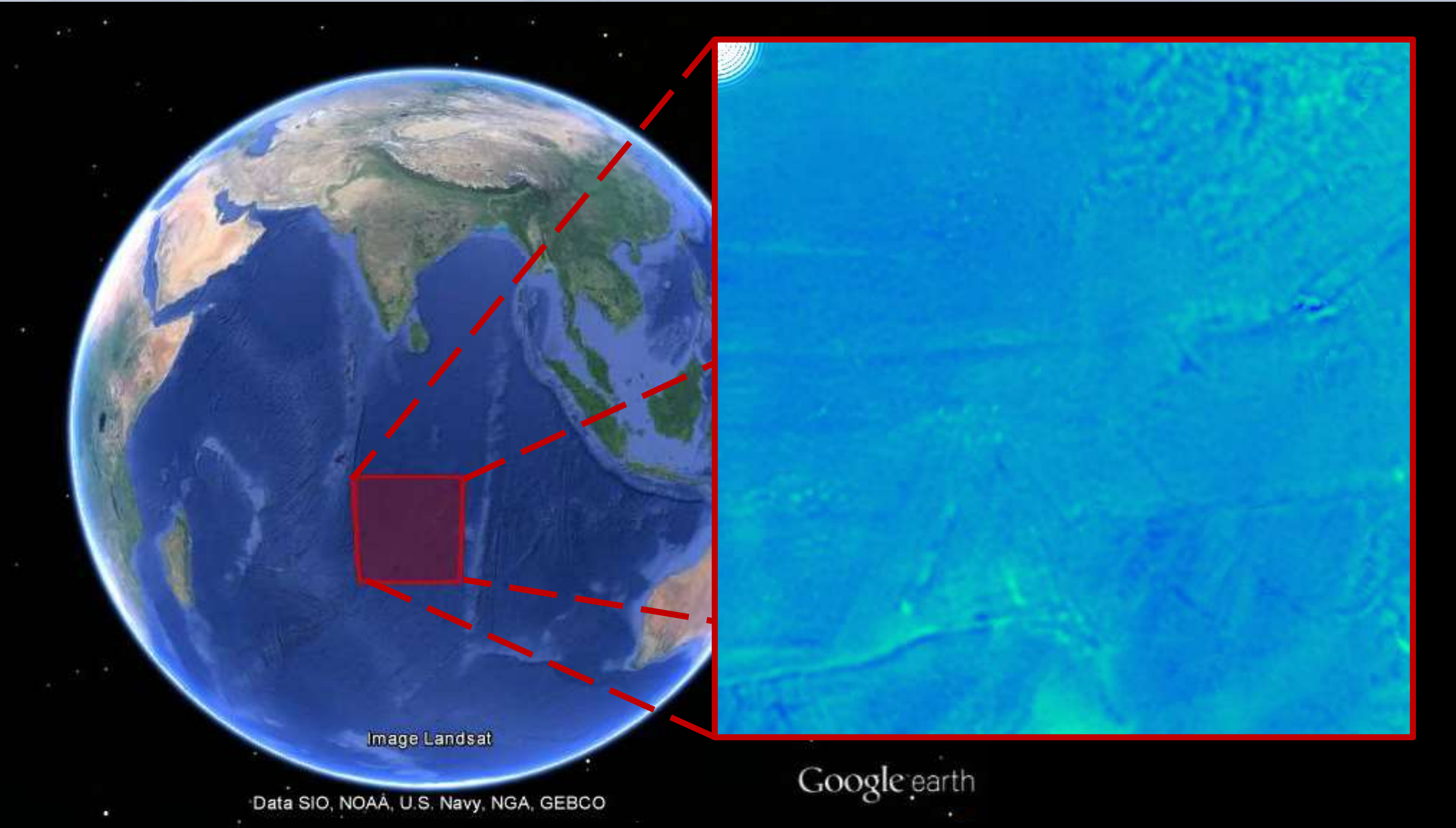
Do tsunami waves show branching?



Do tsunamis show branching?

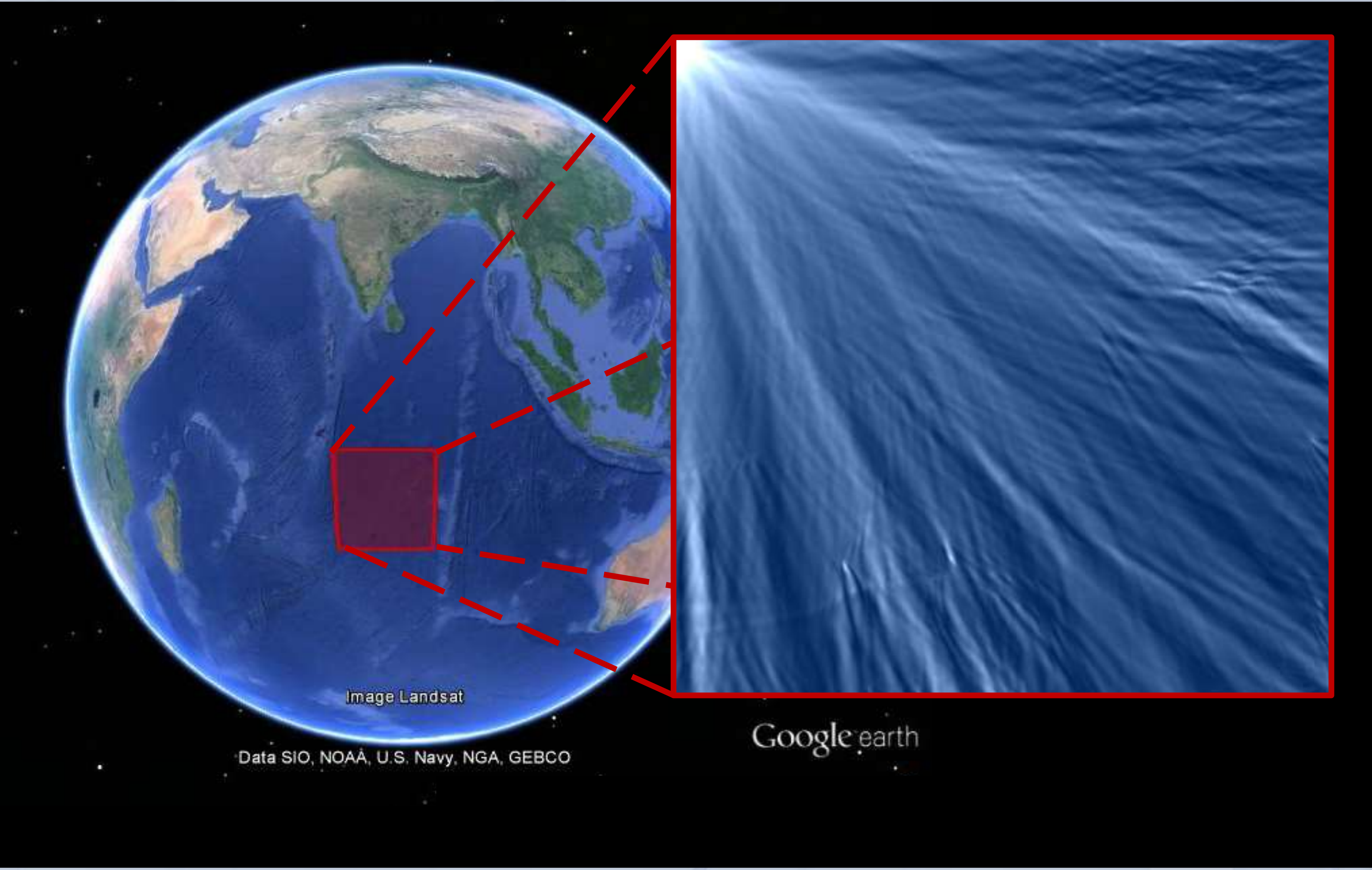


Do tsunamis show branching?

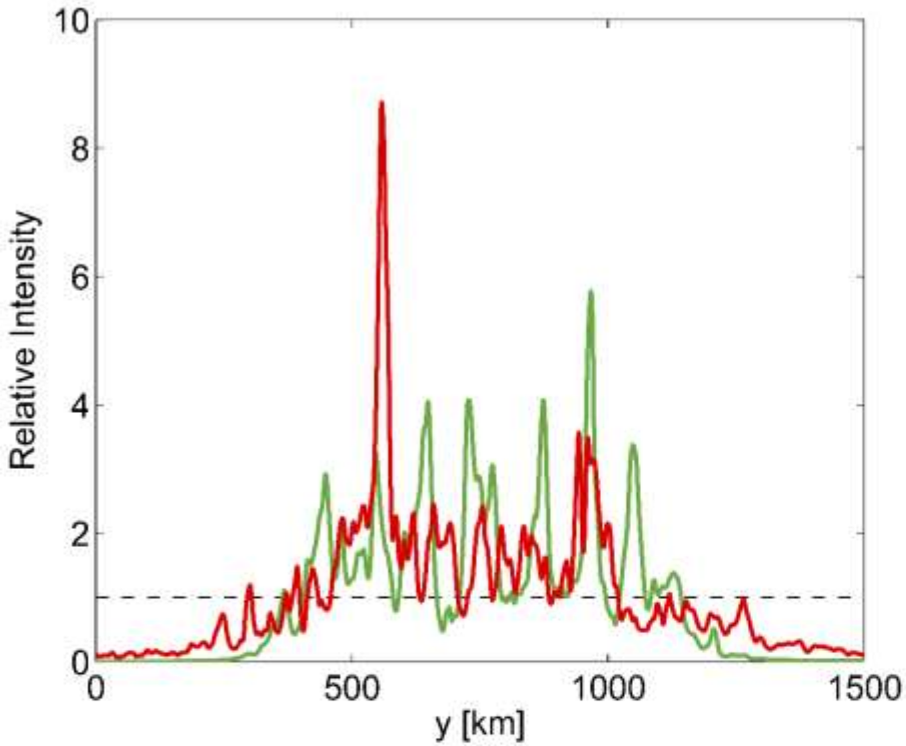


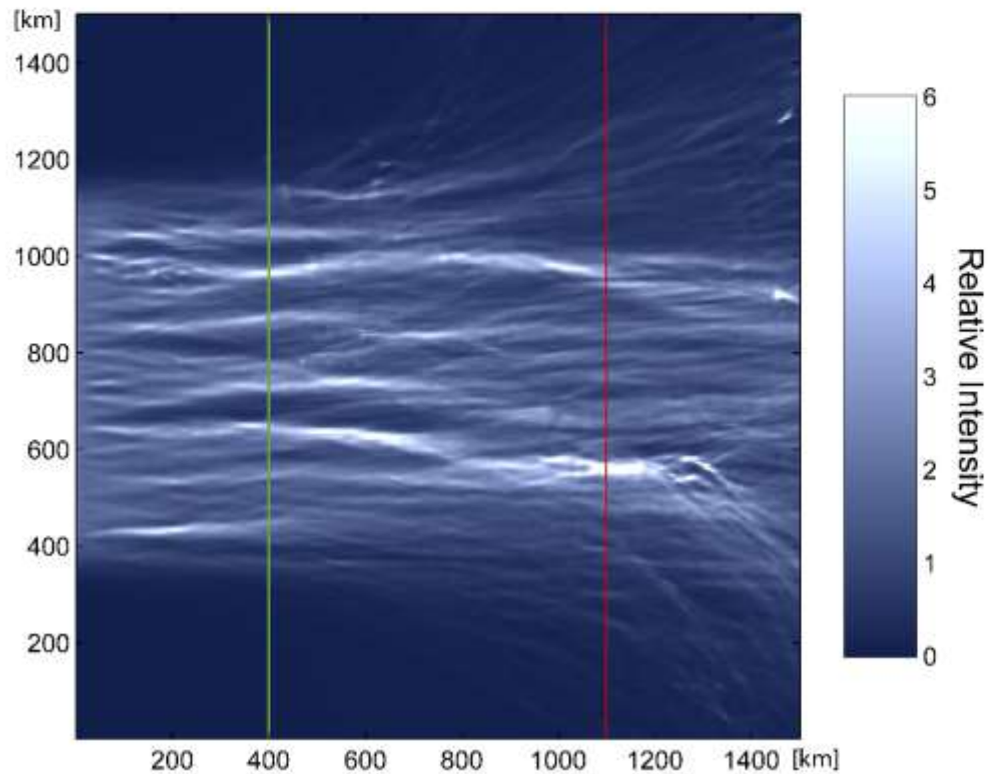
Standard deviation of ocean depth /mean ocean depth ≈ 7%

Do tsunamis show branching?



Tsunami created by a fault source





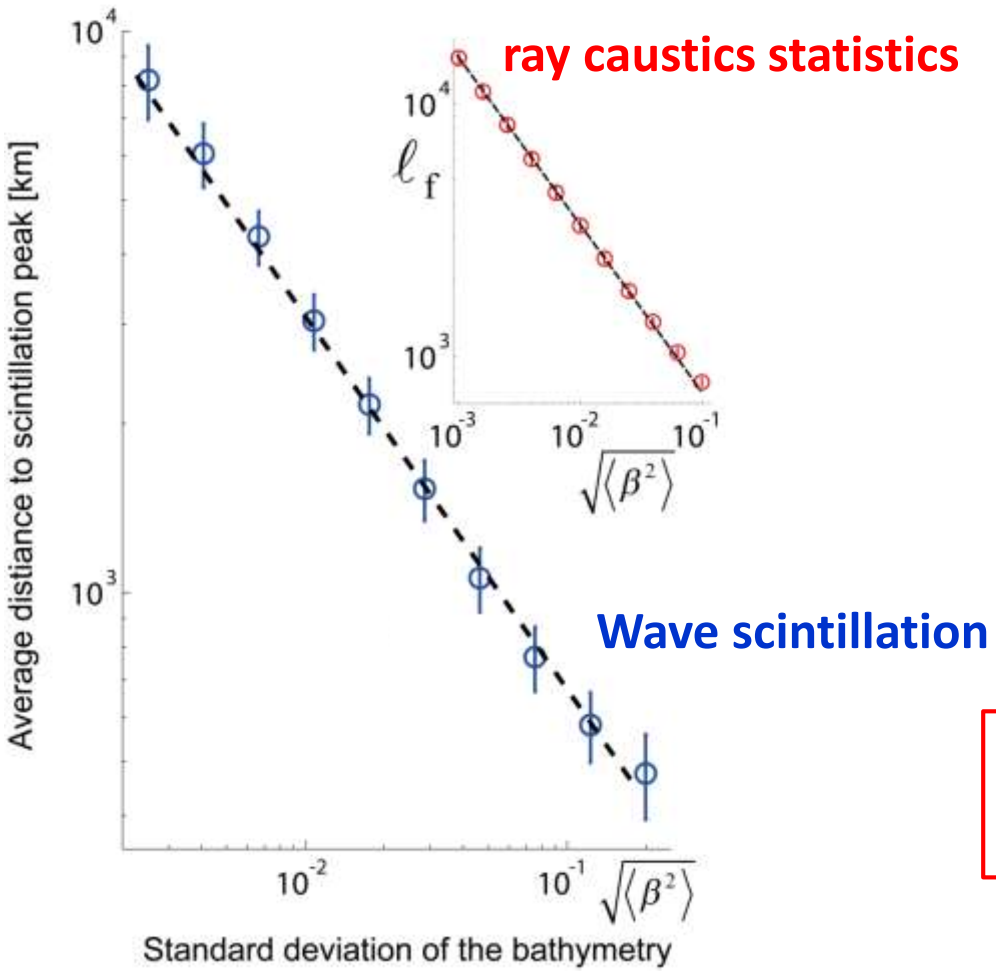
Scaling of shallow water waves height fluctuations

Linearized shallow water wave equations in random bathymetry ß

$$\partial_t^2 \eta(\vec{r},t) = c_0^2 \left(1-\beta(\vec{r})\right) \Delta \eta(\vec{r},t)$$

$$\langle \beta \rangle = 0$$

$$\langle \beta^2 \rangle = \varepsilon^2$$



Ray equations

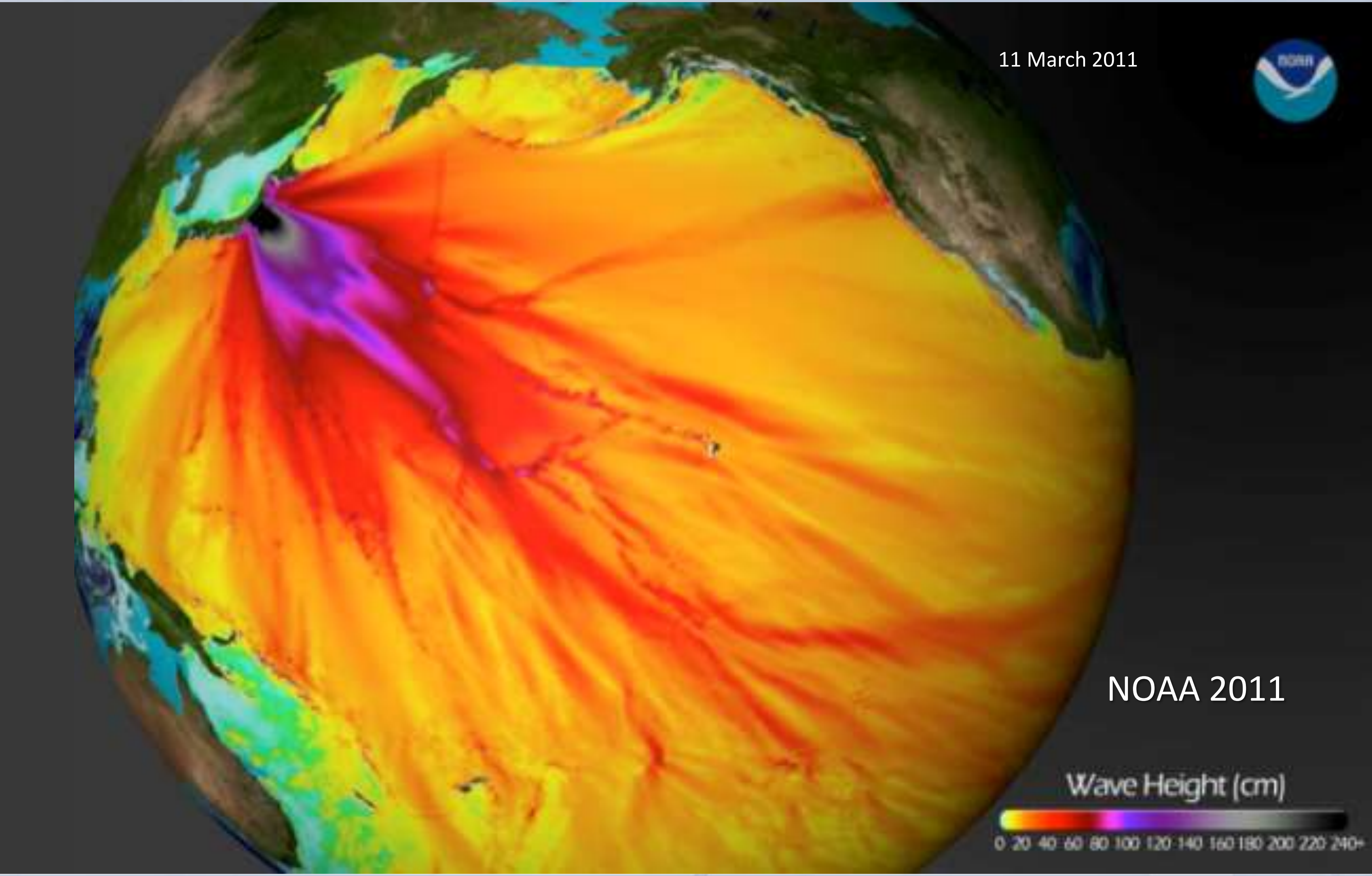
$$\dot{\vec{r}} = c_0^2 \left(1-\beta(\vec{r})\right) \vec{q}$$

$$\dot{\vec{q}} = \frac{\nabla \beta(\vec{r})}{2\left(1-\beta(\vec{r})\right)}$$

$$\boxed{\propto \varepsilon^{-2/3}}$$

Degueldre, Metzger, Geisel, Fleischmann, submitted

Only of academic interest?



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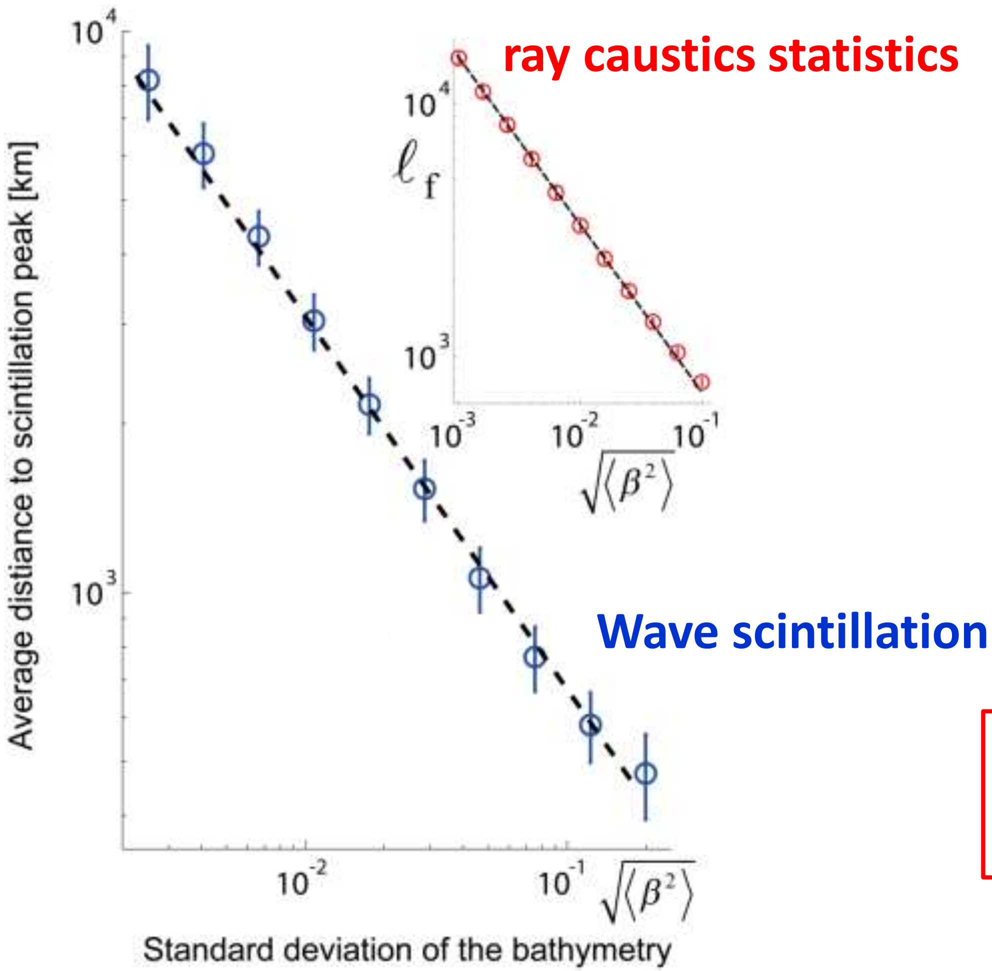

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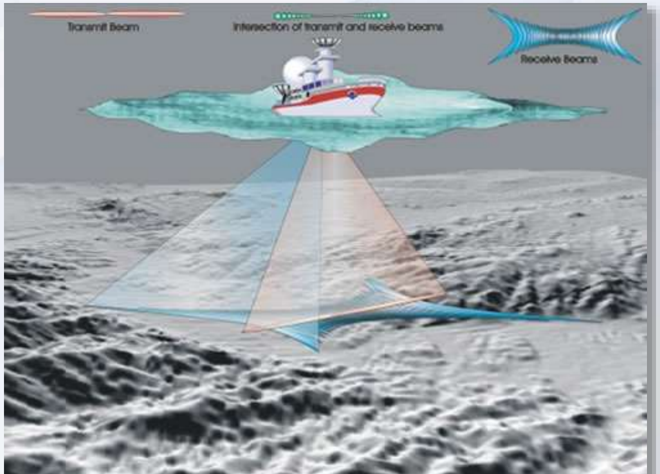
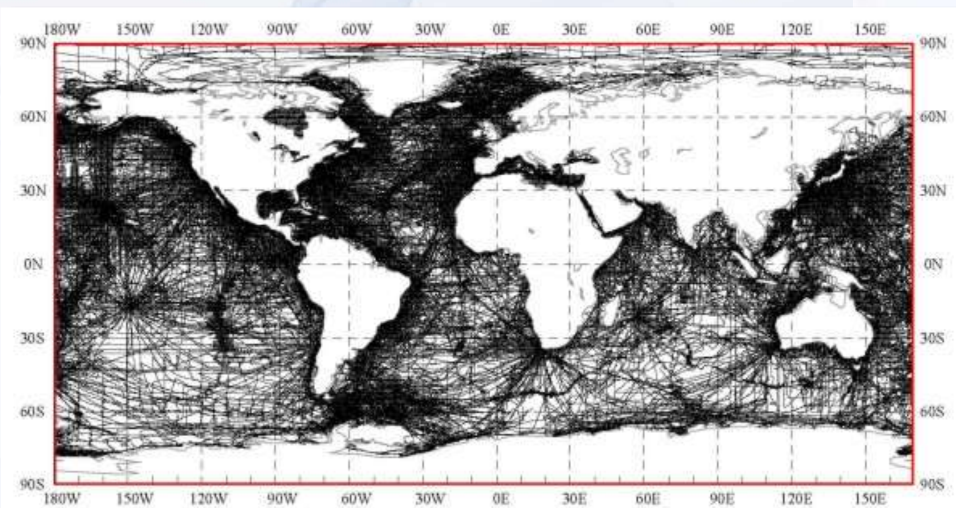
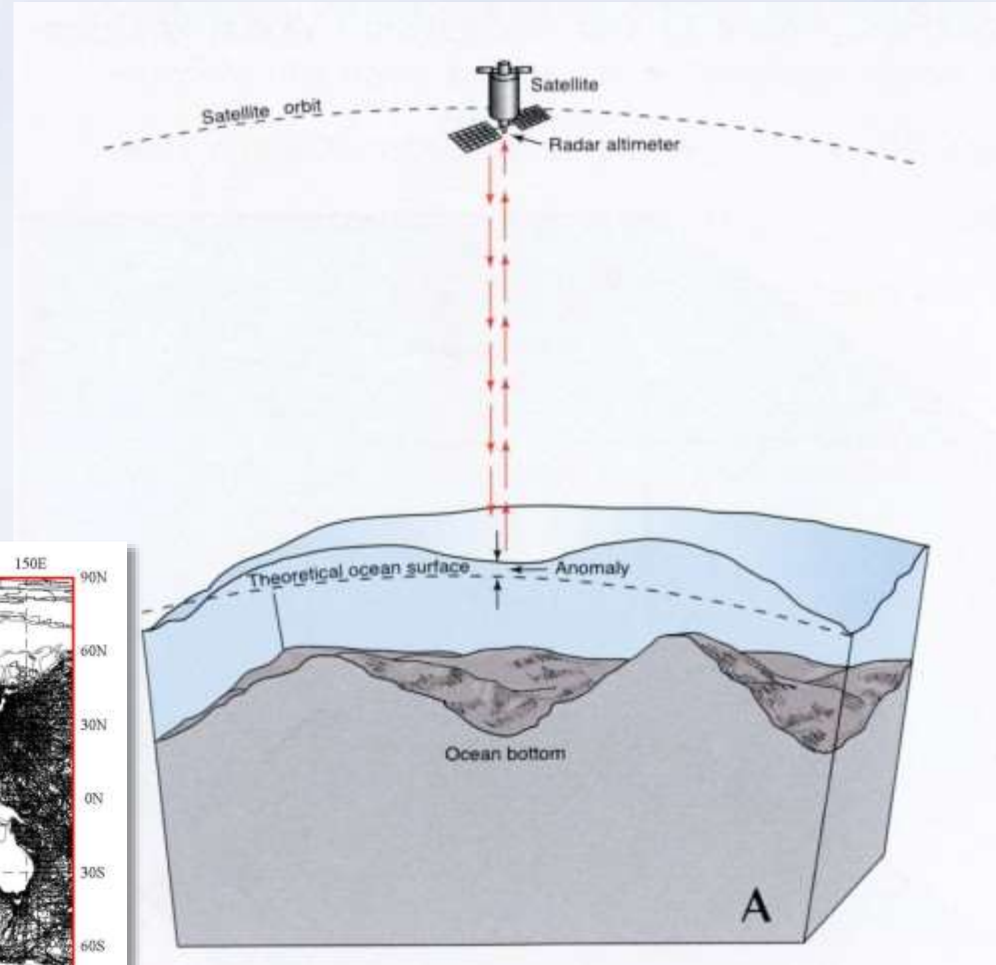
Degueldre, Metzger, Geisel, Fleischmann, submitted

Model of the ocean floor







http://www.gebco.net/general_interest/faq/

Model of the ocean floor

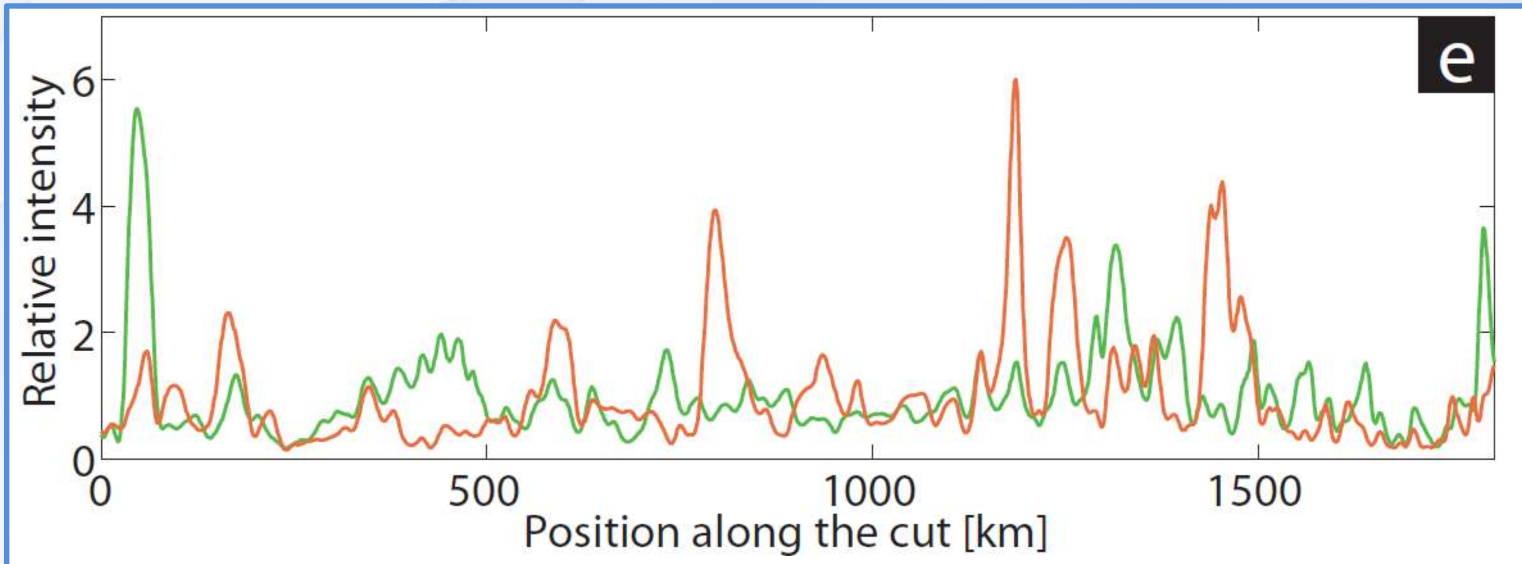
What is the “State-of-the-Art“?

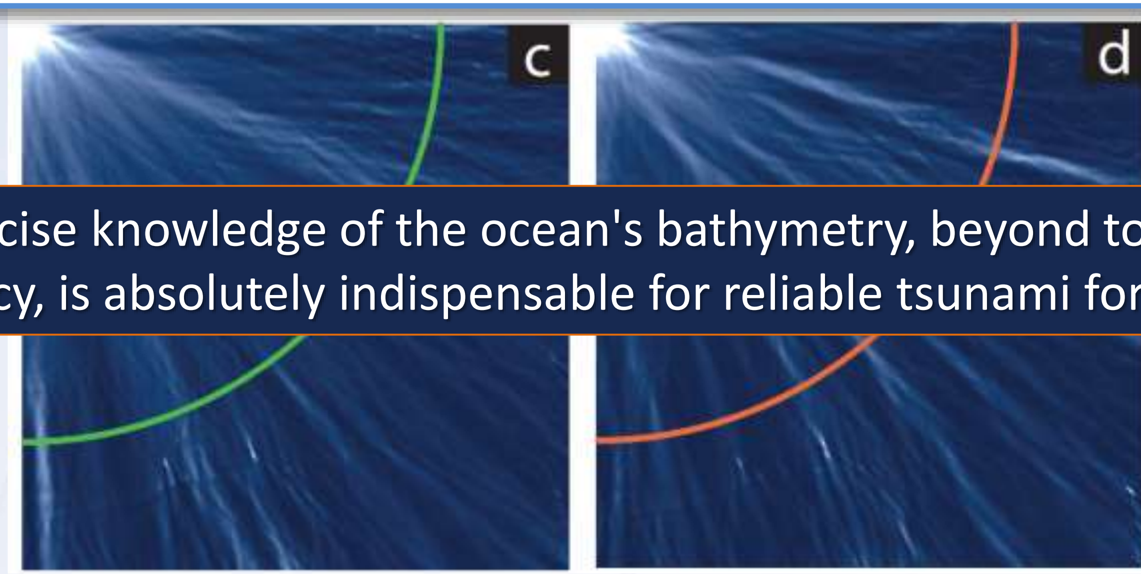
“Topography [...] derived from sparse ship soundings and satellite altimeter measurements reveals the large-scale structures created by seafloor spreading (ridges and transforms) but the horizontal resolution (15 km) and vertical accuracy (250 m) is poor.”

$$\frac{250m}{4\ km} > 6\%$$

Sandwell, D.T., Gille, S.T., and W.H.F. Smith, eds., Bathymetry from Space: Oceanography, Geophysics, and Climate, Geoscience Professional Services, Bethesda, Maryland, June 2002

Add fluctuations to the recorded depth profile < error





A precise knowledge of the ocean's bathymetry, beyond todays accuracy, is absolutely indispensable for reliable tsunami forecasts

In collaboration with...

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External Collaborators

Theory

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Experiment

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Ulrich Kuhl
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(Univs. Marburg & Nice)

DFG
Forschergruppe
FOR760

Conclusions

- General phenomenon of wave propagation in weakly scattering media
- Universal statistics of branches
- Fundamental length scale of transport in disordered systems
- Scaling of peak position in the scintillation index
- Experimental observation of scaling behavior
- Branching leads to heavy-tailed intensity distribution: rogue waves
- Theory of extreme waves
- Branching of tsunami waves
- Tsunami forecasts

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S. Barkhofen, J. J. Metzger, R. Fleischmann, U. Kuhl, and H.-J. Stöckmann, Phys. Rev. Lett. **111**, 183902 (2013).

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H. Dequeldre, J. J. Metzger, T. Geisel, and R. Fleischmann, to appear Nature physics (2015)