Problems of synchronisation in cellular automata: the role of randomness

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Synchronisation as a natural phenomenon

self-synchronisation:

observed in various systems composed of numerous agents

Christaan Huygens
(1629-1695)

self = no external stimulation

examples :

- coupled pendulums,
- fireflies,
- clapping, etc.

H. : longitude problem
"an odd kind of sympathy" (1665)

The cellular automata framework

Space, time and state are discrete.
Does this affect the capacities of "self-organisation"?

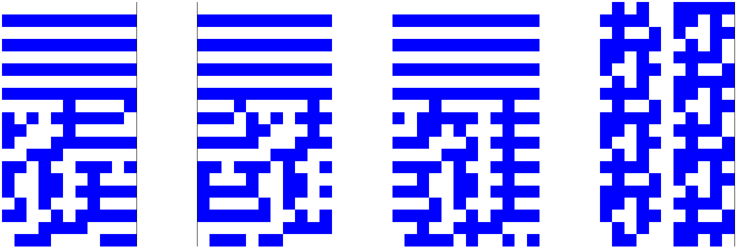
start: from a disordered state
goal: periodic and homogeneous oscillation of cells

CA problem posed by Das, Crutchfield, Mitchell & Hanson:
`Evolving globally synchronized cellular automata`, 1995.

There exists an abundant litterature with continous models... but
$\rightsquigarrow$ what about discrete cellular systems?

Global synchronisation problem

one-dimensional binary systems with periodic boundary conditions.
goal: for **any initial condition**, the system eventually oscillates in the cycle with two homogeneous configurations (**0** and **1**).



Example of rule 1078270911 with radius 2.

A few properties

- A solution is not color-blind, that is, it can not treat the 0s and 1s in the same way.
- A trajectory can not contain two configurations that are in the same class of rotations.
- The height of a solution of size n is bounded by $\chi(n) - 2$, where χ is the number of classes of rotations.

n	1	2	3	4	5	6	7	8	9
$\lvert\chi(n)\rvert$	2	3	4	6	8	14	20	36	60

First step : ECA space

- There exists no ECA which achieves synchronisation.

Proof:

000	001	100	101	010	011	110	111
a	b	c	d	e	f	g	h

a=1 ; h=0 ; d = e = 0 ; etc.

$\rightsquigarrow$ but... difficult to generalise, even to a neighbourhood of size 4

generalisation: formulation as a SAT problem

Formulate our synchronisation problem as a satisfiability problem.

SAT : is there an assignment of boolean variables that makes a given formula true ?
Sat solvers : accept entries in conjunctive normal form (CNF)

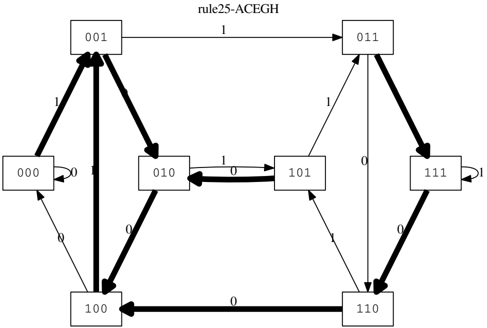
plan : for a given neighb. size, ask for more and more initial conditions to be synchronised until

- either we know that there are no solutions for this neighb.,
- or we find a good candidate to solve the problem for every size

We could tackle neighb. size : $k = 4$, $k = 5$ but not $k = 6$. $\rightsquigarrow$ Is there a proof ?

(im)possibility to solve the problem

idea by G. Richard (Caen university) and M. Redeker:
generate a translating configuration (by $k-1$ cells)



direct:

1001001...
011100100...

or:
de Bruijn diagrams

one auxiliary state is sufficient for solving the problem
or fixed boundary conditions (ECA 3)

The stochastic case

What if we allow the transitions to use randomness ?

000	001	100	101	010	011	110	111
a	b	c	d	e	f	g	h

Now take $a, \ldots, h$ as the probability to update to 1.
we still have $a = 0$ and $h = 0$...

Almost all rules are solutions!

000	001	100	101	010	011	110	111
a	b	c	d	e	f	g	h

- Any rule such that $b, \ldots, g \in (0,1)$ is a solution.

Proof:
$\mathbf{0}-\mathbf{1}$ is the only set of attractive configurations
non-zero probability to reach $\mathbf{0}-\mathbf{1}$ in less than $n/2$ steps.
idea : evolution may follows exactly (Z_t) and reach $\mathbf{0}$ or $\mathbf{1}$.

(Z_t) is the process such that:

$$Z_t(0,0,0) = 1,\; Z_t(1,1,1) = 0,$$
$$Z_t(a,b,c) = \begin{cases} 0 & \text{if } t \text{ is even} \\ 1 & \text{if } t \text{ is odd} \end{cases}$$

example:
00001111000
11100000011
00111111110
10000000000
11111111111

"Efficient" solutions

Measuring efficiency: T(x): synch. time ; random var.
Expected average synch. time : $EAST(n) = 1/2^n \sum_{x \in E_n} T(x)$
Worst expected synch. time : $WEST(n) = \max_{x \in E_n}$

- There are rules for which $WEST(n) = \mathcal{O}(n^2)$

Proof:
consider local rule : take the α-asynchronous shift and invert it.
If we apply an "inversion mask" at each time step, the dynamics is the same as the α-asynchronous shift

$\rightsquigarrow$ Does there exist (qualitatively) more efficient rules?

To sum up

deterministic case:
"good rules" only solve the problem "statistically"
There exists no perfect deterministic solution

What are the max. synch. size for a given neighbourhood?
(SAT solvers gives only partial results)

- Huge gap with stochastic CA: noise makes it "easy" to solve the problem !

exploration of the stochastic solutions is an open problem

Questions

For a given neighbourhood, what can we say about the maximum synchronisation size ?

How can we go further with the SAT formulation?

Are there stochastic solutions with a linear convergence time ?

In what other areas can the use of randomness appear as an aid ? density classification problem, consensus, etc.

On the role of randomness

Philosophical Transactions of the Royal Society of London
Series B, Biological Sciences, Vol. 237, 1952, pp. 37-72.

THE CHEMICAL BASIS OF MORPHOGENESIS

By A. M. TURING, F.R.S. *University of Manchester*

(*Received* 9 *November* 1951—*Revised* 15 *March* 1952)

It is suggested that a system of chemical substances, called morphogens, reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis. Such a system, although it may originally be quite homogeneous, may later develop a pattern or structure due to an instability of the homogeneous equilibrium, which is triggered off by random disturbances. Such reaction-diffusion systems are considered in some detail in the case of an isolated ring of cells, a mathematically convenient, though biologically unusual system. The investigation is chiefly concerned with the onset of instability. It is found that there are six essentially different forms which this may take. In the most interesting form stationary waves appear on the ring. It is suggested that this might account, for instance, for the tentacle patterns on *Hydra* and for whorled leaves. A system of reactions and diffusion on a sphere is also considered. Such a system appears to account for gastrulation. Another reaction system in two dimensions gives rise to patterns reminiscent of dappling. It is also suggested that stationary waves in two dimensions could account for the phenomena of phyllotaxis.

The purpose of this paper is to discuss a possible mechanism by which the genes of a zygote may determine the anatomical structure of the resulting organism. The theory does not make any new hypotheses; it merely suggests that certain well-known physical laws are sufficient to account for many of the facts. The full understanding of the paper requires a good knowledge of mathe-

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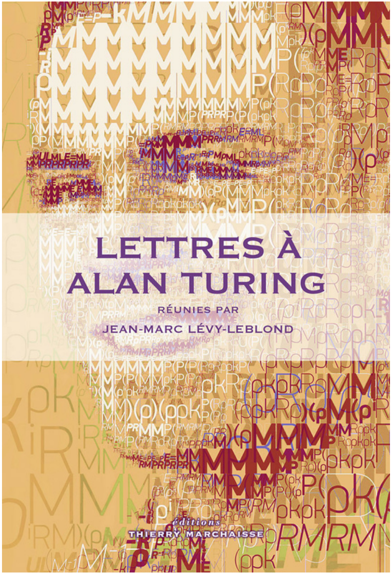
Turing



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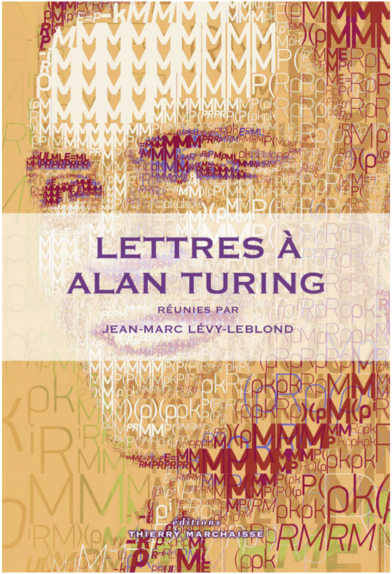
Turing



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Danke schön !

SAT coding (example for the ECA case)

Two types of boolean variables :

the transition rule :

000	001	100	101	010	011	110	111
t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7

evolution of initial condition:

```
abcdefgh
ijklmnop
qrstuvwx
```

- Blinking condition :

$F_{\mathrm{b}} = t_0 \wedge \bar{t}_7.$

- Coding of the initial condition $x = 00101011 \leftrightarrow \mathtt{abcdefgh}$:

$F_{\mathrm{ic}}(x) = \bar{a} \wedge \bar{b} \wedge c \cdots \wedge h$

- Synchronisation condition:

$F_{\mathrm{synch}}(x) = (q = r) \wedge (r = s) \wedge \cdots \wedge (w = x)$

Consistency conditions

how do we code the coherence of evolution and transition table ?

```
00101011 abcdefgh
00010111 ijklmnop
```

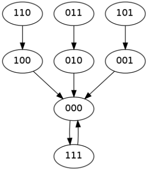
For $t = 0, i = 4$, write : $F_{\mathrm{cons}}(x, t, i) = c \wedge \bar{d} \wedge e \wedge (l = t_5)$
transformed into: $F_{\mathrm{cons}}(x, t, i) = c \wedge \bar{d} \wedge e \wedge (l \vee \bar{t_5}) \wedge (\bar{l} \vee t_5)$

but next state not known, there are 8 possibilities for each local transition...

see article AUTOMATA'15 for details

a few esults

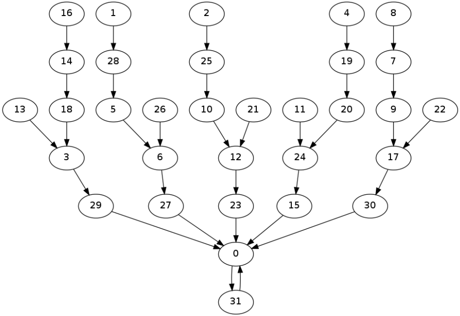
ECA case : maximum synch. size is 3
e.g. rule 9



$k = 4$ case:
maximum synch. size is 6.

- rules of height 4,5:
  e.g. (5419,11095),
- 1 rule of height 6:
  (4363,12151).

e.g. rule 5419



$k = 5$ case : impossibility to finish the exploration ! (4.10^9 rules)

Solution with an auxiliary state

local rule:

$$f_{\mathrm{A}}(x,y,z)=\begin{cases}\mathtt{0} & \text{if } (x,y,z)=(1,1,1) \text{ or } x=\mathtt{A} \text{ or } y=\mathtt{A} \text{ or } z=\mathtt{A}\\ \mathtt{1} & \text{if } (x,y,z)=(0,0,0)\\ \mathtt{A} & \text{otherwise.}\end{cases}$$

example:

```
00000000000000000000000    t = 4
XXXXXXXXXXXXXXXXXAAAAXX    t = 3
000000000000000000XX000    t = 2
XXAA0AAAAAXAAAXAA0000AA    t = 1
000XXX00X000X000XXXXXX0    t = 0
```

[back]